22.01.2015

Exam on Quantum Field Theory I

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Problem 1 (5 points)

Consider a free real scalar field with the following action

$$S = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \right). \tag{1}$$

a) Derive the equation of motion for ϕ using the variational principle.

Hint: To do this, substitute $\phi \to \phi + \delta\phi$, expand S to first order in the small quantity $\delta\varphi$ and impose $\delta S = 0$.

We vary S with respect to ϕ and $\partial_\mu \phi$ whereby the boundaries are held fixed

$$\delta \phi|_{\text{boundary}} = 0 \qquad \delta \partial_\mu \phi|_{\text{boundary}} = 0$$

Varying ϕ by $\phi \to \phi' = \phi + \delta\phi$ transforms the action

$$S \to S' = S + \delta S = \int d^4x \left(\frac{1}{2} \partial_\mu (\phi + \delta\phi) \partial^\mu (\phi + \delta\phi) - \frac{1}{2} m^2 (\phi + \delta\phi)^2 \right)$$

$$= \int d^4x \left(\frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi + \frac{1}{2} \partial_\mu \phi \, \partial^\mu \delta\phi + \frac{1}{2} \partial_\mu \delta\phi \, \partial^\mu \phi + \frac{1}{2} \partial_\mu \delta\phi \, \partial^\mu \delta\phi \right.$$

$$\left. - \frac{1}{2} m^2 \phi^2 - m^2 \phi \, \delta\phi - \frac{1}{2} m^2 \delta\phi^2 \right)$$

$$= S + \int d^4x \left(\partial_\mu \phi \, \partial^\mu \delta\phi + \mathcal{O}((\partial \delta\phi)^2) - m^2 \phi \, \delta\phi + \mathcal{O}(\delta\phi^2) \right)$$

$$0 \stackrel{!}{=} \delta S \approx \int d^4x \left(\partial_\mu \phi \, \partial^\mu \delta\phi - m^2 \phi \, \delta\phi \right) = \underbrace{(\partial_\mu \phi) \, \delta\phi \Big|_{\text{boundary}}}_{0} + \int d^4x \left(-(\partial_\mu \partial^\mu \phi) \delta\phi - m^2 \phi \, \delta\phi \right)$$

$$= - \int d^4x \left(\partial^2 \phi + m^2 \phi \right) \delta\phi$$

Now since $\delta\phi$ is not zero along all integration paths, we can infer that for $\delta S = 0$ to be true in general necessarily

$$\partial^2 \phi + m^2 \phi = (\partial^2 + m^2) \phi = 0$$

b) Consider the operator

$$\phi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2p_0}} \left(a_p^{+} e^{ipx} + a_p e^{-ipx} \right) \Big|_{p_0 = \sqrt{p^2+m^2}} \qquad (2)$$

Show that (2) satisfies the equation of motion $(\Box + m^2)\phi(x) = 0$.

$$(\Box + m^2)\phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2p_0}} (\partial^2 + m^2)\left(a_p^{+} e^{ipx} + a_p e^{-ipx} \right)$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2p_0}} \underbrace{(-p^2 + m^2)}_{0} \left(a_p^{+} e^{ipx} + a_p e^{-ipx} \right) = 0$$

c) Calculate from (1), the Lagrangian, the canonical conjugate momentum density, $\pi(x)$, to $\phi(x)$. Show that $\pi(x) = \dot{\phi}(x)$ and use (2) to write down $\pi(x)$ in a form analogous to (2).

$$\pi(x) = \frac{\partial \mathcal{L}(\phi(x), \partial\phi(x))}{\partial(\dot{\phi}(x))} = \frac{\partial}{\partial(\dot{\phi}(x))}\left(\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 \right) = \frac{\partial}{\partial(\dot{\phi}(x))}\left(\frac{1}{2}\dot{\phi}^2 - \frac{1}{2}\Delta\phi^2 - \frac{1}{2}m^2\phi^2 \right)$$

$$= \dot{\phi}(x)$$

$$\pi(x) = \frac{\partial}{\partial t} \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2p_0}} \left(a_p^{+} e^{ipx} + a_p e^{-ipx} \right) = i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{p_0}{2}} \left(a_p^{+} e^{ipx} - a_p e^{-ipx} \right)$$

d) Propose commutation relations for a_p and a_p^{+} such that π and ϕ fulfil the canonical commutation relations. Verify your ansatz by explicitly evaluating the commutators

$$[\pi(\vec{x},t), \pi(\vec{y},t)] = [\phi(\vec{x},t), \phi(\vec{y},t)] = 0 \quad \text{and} \quad [\phi(\vec{x},t), \pi(\vec{y},t)] = i\delta^{(3)}(\vec{x}-\vec{y})$$

Proposition: $[a_p, a_q] = [a_p^{+}, a_q^{+}] = 0$ and $[a_p, a_q^{+}] = (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q})$

Verification:

$$[\pi(\vec{x},t), \pi(\vec{y},t)] = i \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{p_0}{2}}\, e^{ipx}\, i \int \frac{d^3q}{(2\pi)^3} \sqrt{\frac{q_0}{2}}\, e^{iqy}\, [a_p^{+} - a_p, a_q^{+} - a_q] = 0,$$

since $[a_p^{+} - a_p, a_q^{+} - a_q] = \underbrace{[a_p^{+}, a_q^{+}]}_{0} - [a_p^{+}, a_q] - [a_p, a_q^{+}] + \underbrace{[a_p, a_q]}_{0}$

$$= (2\pi)^3 \delta^{(3)}(\vec{q}-\vec{p}) - (2\pi)^3 \delta^{(3)}(\vec{p}-\vec{q}) = 0$$

$$[\phi(\vec{x},t),\phi(\vec{y},t)] = \int \frac{d^3p}{(2\pi)^3}\frac{e^{ipx}}{\sqrt{2p_0}}\int\frac{d^3q}{(2\pi)^3}\frac{e^{iqy}}{\sqrt{2q_0}}[a_p^\dagger + a_{-p}, a_q^\dagger + a_{-q}] = 0,$$

since $$[a_p^\dagger + a_{-p}, a_q^\dagger + a_{-q}] = \underbrace{[a_p^\dagger, a_q^\dagger]}_{0} + [a_p^\dagger, a_{-q}] + [a_{-p}, a_q^\dagger] + \underbrace{[a_{-p}, a_{-q}]}_{0}$$

$$= -(2\pi)^3\delta^{(3)}(-\vec{q}-\vec{p}) + (2\pi)^3\delta^{(3)}(-\vec{p}-\vec{q}) = 0$$

$$[\phi(\vec{x},t),\pi(\vec{y},t)] = \int\frac{d^3p}{(2\pi)^3}\frac{e^{ipx}}{\sqrt{2p_0}}\, i\int\frac{d^3q}{(2\pi)^3}\sqrt{\frac{q_0}{2}}\, e^{iqy}[a_p^\dagger + a_{-p}, a_q^\dagger - a_{-q}]$$

$$\left[[a_p^\dagger + a_{-p}, a_q^\dagger - a_{-q}] = -[a_p^\dagger, a_{-q}] + [a_{-p}, a_q^\dagger] = (2\pi)^3\delta^{(3)}(-\vec{q}-\vec{p}) + (2\pi)^3\delta^{(3)}(-\vec{p}-\vec{q})\right]$$

$$= i\int\frac{d^3p}{(2\pi)^3}\frac{e^{ipx}}{\sqrt{2p_0}}\int\frac{d^3q}{(2\pi)^3}\sqrt{\frac{q_0}{2}}\, e^{iqy}\, 2(2\pi)^3\delta^{(3)}(-\vec{p}-\vec{q})$$

$$= i\int\frac{d^3p}{(2\pi)^3} e^{ipx}e^{-ipy} = i\delta^{(3)}(\vec{x}-\vec{y})$$

<u>Problem 2 (5 points)</u>

The fermionic part of the QED Lagrangian density is given by

$$\mathcal{L} = \bar{\Psi}(i\not{\partial} - m)\Psi - e\bar{\Psi}\not{A}\Psi. \tag{3}$$

a) Using the Clifford algebra in $D=4$, calculate $\gamma^\mu\gamma_\mu$ and $\gamma^\mu\gamma^\rho\gamma_\mu$.

The Clifford algebra Cliff(1,3) is defined as the algebra spanned by $n\times n$-matrices $(\gamma^\mu)^A{}_B$, $\mu\in\{0,1,2,3\}$ and $A,B\in\{1,\dots,n\}$ such that

$$\{\gamma^\mu,\gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\eta^{\mu\nu}\mathbb{1}.$$

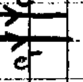
Therefore, $$\gamma^\mu\gamma^\nu = \begin{cases}\eta^{\mu\nu}\mathbb{1} & \text{for } \mu=\nu\\ -\gamma^\nu\gamma^\mu & \text{for } \mu\neq\nu\end{cases} \quad\text{and}\quad \begin{cases}(\gamma^0)^2 = \gamma^0\gamma^0 = \eta^{00}\mathbb{1} = \mathbb{1}\\ (\gamma^i)^2 = \gamma^i\gamma^i = \eta^{ii}\mathbb{1} = -\mathbb{1}\end{cases}$$

$$\gamma^\mu\gamma_\mu = (\gamma^0)^2 - (\gamma^1)^2 - (\gamma^2)^2 - (\gamma^3)^2 = \mathbb{1} - (-\mathbb{1}) - (-\mathbb{1}) - (-\mathbb{1}) = 4\cdot\mathbb{1}$$

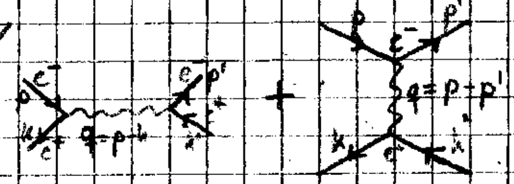
$$\gamma^\mu\gamma^\rho\gamma_\mu = (2\eta^{\mu\rho}\mathbb{1} - \gamma^\rho\gamma^\mu)\gamma_\mu = 2\eta^{\mu\rho}\gamma_\mu\mathbb{1} - \gamma^\rho\gamma^\mu\gamma_\mu$$

$$= 2\gamma^\rho - 4\gamma^\rho = -2\gamma^\rho$$

b) Draw the Feynman graphs for the non-trivial scattering amplitude $e^+ + e^- \rightarrow e^+ + e^-$ to order e^2.

Non-trivial scattering excludes the term δ_{fi} from the S-matrix element $\langle f | S | i \rangle$ represented by the diagram  (order e^0).

Order e^1: /

Order e^2: 

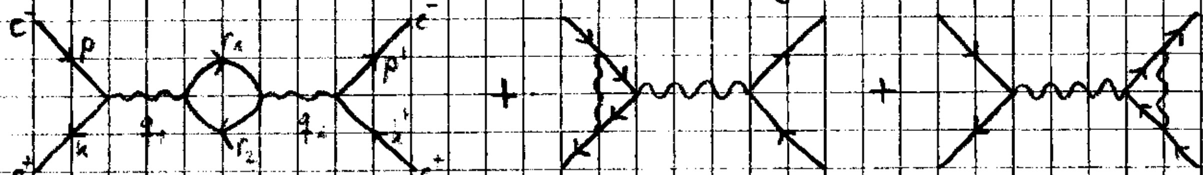
c) Write down the invariant matrix element $i\mathcal{M}$ for this process.

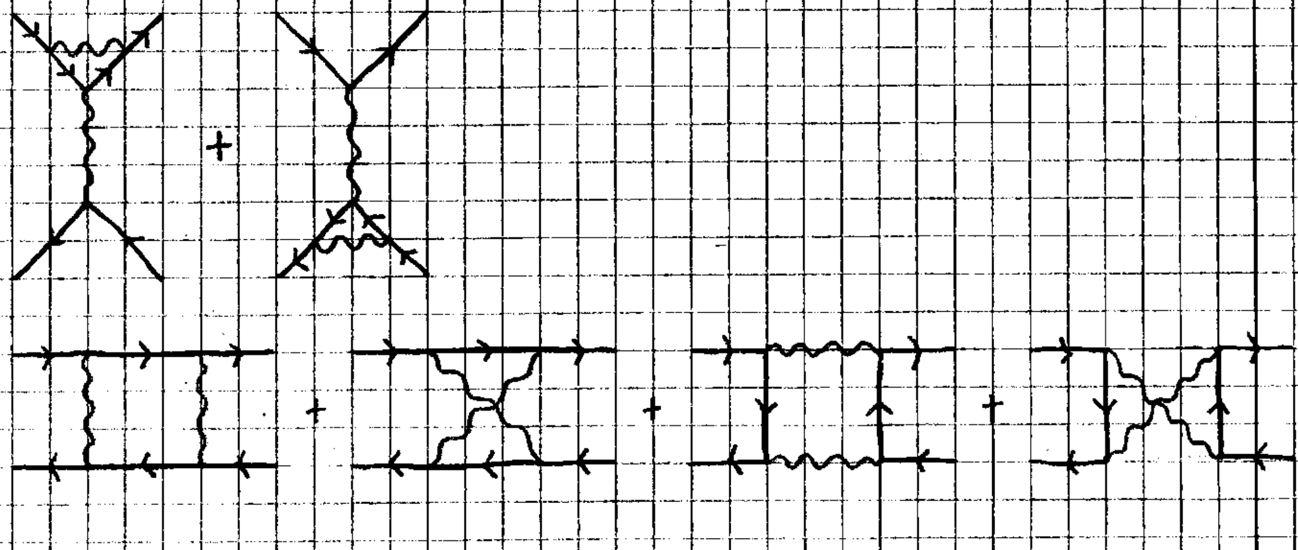
$$i\mathcal{M} = i\,\bar{u}_s(p')\,v_r(k')\,(-ie\gamma^\nu)\left(\frac{-i\eta_{\mu\nu}}{(p-k)^2} + \frac{-i\eta_{\mu\nu}}{(p-p')^2}\right)(-ie\gamma^\mu)\,u_s(p)\,\bar{v}_r(k)$$

$$= -e^2\,\bar{u}_s(p')\,v_r(k')\,\gamma^\mu\eta_{\mu\nu}\gamma^\nu\left(\frac{1}{(p-k)^2} + \frac{1}{(p-p')^2}\right)u_s(p)\,\bar{v}_r(k)$$

$$= -4e^2\,\bar{u}_s(p')\,v_r(k')\left(\frac{1}{(p-k)^2} + \frac{1}{(p-p')^2}\right)u_s(p)\,\bar{v}_r(k)$$

d) As b) but this time the Feynman diagrams of order e^4.





Problem 3 (5 points)

For the $\lambda\phi^4$-theory the propagator is given by

$$\frac{i}{p^2-m_0^2-\Pi(p^2)}=\frac{iZ}{p^2-m^2}+\int_{m_t^2}^{\infty}dm'^2\,\sigma(m'^2)\frac{i}{p^2-m'^2} \tag{4}$$

with physical mass m, Lagrangian mass parameter m_0, self-energy $\Pi(p^2)$ and the two-particle threshold m_t.

a) From the position of the 1-particle pole in this equation, deduce a relation between m_0 and m.

Not covered in our treatment of QFT!

b) By comparing the residues at the 1-particle poles, deduce a relation between Z and m^2.

Hint: Use the fact that the integral on the right hand side of (4) has no poles near $p^2=m^2$. Perform a first-order Taylor expansion of $\Pi(p^2)$ at $p^2=m^2$.

Not covered either by Berges' QFT lecture!

Problem 4 (5 points)

The Lagrangian density of scalar QED is given by

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)^\dagger D^\mu \phi + m^2 \phi^\dagger \phi \qquad (5)$$

with $D_\mu = \partial_\mu + ie A_\mu$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$.

a) Show that the action (5) is invariant under $\phi \to e^{ie\theta}\phi$ with $\theta \in \mathbb{R}$.

$$\mathcal{L} \to \mathcal{L}' = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu e^{ie\theta}\phi)^\dagger D^\mu e^{ie\theta}\phi + m^2 (e^{ie\theta}\phi)^\dagger e^{ie\theta}\phi$$

$$= -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + (D_\mu \phi)\underbrace{e^{-ie\theta} e^{ie\theta}}_{1} D_\mu \phi + m^2 \phi^\dagger \underbrace{e^{-ie\theta} e^{ie\theta}}_{1}\phi = \mathcal{L}$$

b) Calculate the Noether current and charge corresponding to the symmetry transformation of a).

For the continuous symmetry in a), we can write infinitesimally

$$\phi \to \phi' = e^{ie\theta}\phi = (1 + ie\theta + \sigma(\theta^2))\phi = \phi + \theta \underbrace{ie\phi}_{\delta\phi} - \sigma(\theta^2)$$

Furthermore, from $\mathcal{L} = \mathcal{L}'$ we can deduce that

$$\mathcal{L} \to \mathcal{L}' = \mathcal{L} + \underbrace{\theta\delta\mathcal{L}}_{0} + \underbrace{\sigma(\theta^2)}_{0}, \quad \text{i.e. since } \delta\mathcal{L} = \partial_\mu F^\mu, \ F^\mu = 0$$

Then, using the definition of the Noether current, we find

$$j^\mu := \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^\dagger)} \delta\phi^\dagger - F^\mu = (D^\mu \phi)^\dagger ie\phi + (D^\mu \phi)(-ie\phi^\dagger)$$

From this, the Noether charge can be calculated by integrating the current's zero component over space

$$Q = \int d^3x \, j^0(t, \vec{x}) = \int d^3x \, ie \left(\phi (D^0 \phi)^\dagger - \phi^\dagger D^0 \phi\right)$$

$$= ie \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \int d^3x \left(e^{-ipx}(a_p + b_{-p}^\dagger)(D^0 e^{-iqx}(a_q + b_{-q}^\dagger)) - e^{ipx}(a_p^\dagger + b_{-p}) D^0 e^{-iqx}(a_q + b_q^\dagger) \right)$$

$$= ie \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}} \int d^3x \Big(e^{-ipx}(a_p + b_p^\dagger)(iE_q - ieA^0) e^{iqx}(a_q^\dagger + b_q)$$

$$- e^{ipx}(a_p^\dagger + b_p)(-iE_q + ieA^0) e^{-iqx}(a_q + b_q^\dagger) \Big)$$

$$= -e\int\frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\int\frac{d^3q}{(2\pi)^3}\frac{1}{\sqrt{2E_q}}\Big((2\pi)^3\delta^{(3)}(\vec{q}-\vec{p})\,e^{i(E_q-E_p)t}(E_q-eA^0)(a_p+b^\dagger_{-p})(a^\dagger_q+b_{-q})$$

$$+(2\pi)^3\delta^{(3)}(\vec{p}-\vec{q})\,e^{i(E_p-E_q)t}(E_q-eA^0)(a^\dagger_p+b_{-p})(a_q+b^\dagger_{-q})\Big)$$

$$= -e\int\frac{d^3p}{(2\pi)^3}\frac{1}{2E_p}\Big((E_p-eA^0)(a_p+b^\dagger_{-p})(a^\dagger_p+b_{-p})+(E_p-eA^0)(a^\dagger_p+b_{-p})(a_p+b^\dagger_{-p})\Big)$$

$$= -e\int\frac{d^3p}{(2\pi)^3}\Big(\frac{1}{2}-\frac{eA^0}{2E_p}\Big)\Big(a_pa^\dagger_p+a_pb_{-p}+b^\dagger_{-p}a^\dagger_p+b^\dagger_{-p}b_{-p}+a^\dagger_pa_p+a^\dagger_pb^\dagger_{-p}+b_{-p}a_p+b_{-p}b^\dagger_{-p}\Big)$$

$$= -e\int\frac{d^3p}{(2\pi)^3}\Big(\frac{1}{2}-\frac{eA^0}{2E_p}\Big)\Big(2a^\dagger_pa_p+(2\pi)^3\delta^{(3)}(\vec{p}-\vec{p})+2a_pb_{-p}+2a^\dagger_pb^\dagger_{-p}+2b^\dagger_pb_p+(2\pi)^3\delta(\vec{p}-\vec{p})\Big)$$

$$= -e\int\frac{d^3p}{(2\pi)^3}\Big(1-\frac{eA^0}{E_p}\Big)\Big(a^\dagger_pa_p+b^\dagger_pb_p+a_pb_{-p}+a^\dagger_pb^\dagger_{-p}\Big)-e\int d^3p\Big(1-\frac{eA^0}{E_p}\Big)\delta^{(3)}(\vec{0})$$

c) Expand the Lagrangian in terms of the interaction terms and state the Feynman rules for the vertices of scalar QED.
Hint: What does a derivative ∂^μ correspond to in momentum space?

The only term in (5) representing interaction is the middle one.

$$(D_\mu\phi)^\dagger D^\mu\phi = (\partial_\mu - ieA_\mu)\phi^\dagger(\partial^\mu+ieA^\mu)\phi$$

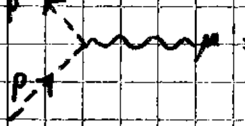
$$= \partial_\mu\phi^\dagger\partial^\mu\phi + (\partial_\mu\phi^\dagger)ieA^\mu\phi - ieA_\mu\phi^\dagger\partial^\mu\phi + e^2A_\mu\phi^\dagger A^\mu\phi$$

$$= \partial_\mu\phi^\dagger\partial^\mu\phi + ie(\phi\,\partial_\mu\phi^\dagger - \phi^\dagger\partial_\mu\phi)A^\mu + e^2A_\mu A^\mu\phi^\dagger\phi$$

Therefore, the interaction Lagrangian reads

$$\mathcal{L}_{int} = ie(\phi\,\partial_\mu\phi^\dagger - \phi^\dagger\partial_\mu\phi)A^\mu + e^2A_\mu A^\mu\phi^\dagger\phi$$

The Feynman rules for the vertices of scalar QED are



$= -ie(p+p')^\mu$

$= 2ie^2\eta^{\mu\nu}$

Problem 5 (5 points)

a) Describe how the Ward-Takashi identity guarantees that no negative norm states are created in interactions for a massive vector field.

Same as 3 a) and b), was not discussed in this lecture!

b) Explain how one can see that the Dirac spinor representation of the Lorentz group is reducible in four dimensions.

An important concept in representation theory is that of an irreducible representation (irrep): Its representation space does not split into a direct sum of two vector spaces in a manner respected by the group action, i.e. it is not possible to find a basic of the representation space in which

$$R(\Lambda) = \left(\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right) \quad \forall \Lambda, \text{ where } R \text{ is an automorphism and } \Lambda \text{ a Lorentz transf.}$$

The Dirac spinor representation of Cliff(1,3) is not irreducible as a representation of Spin(1,3) since the Dirac matrices can be chosen as

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

in which case ψ^A transforms under a spatial rotation and Lorentz boost ($\omega_{oj} = \chi_j$) as

$$\psi \longrightarrow \begin{pmatrix} e^{-\frac{i}{2}\alpha \vec{n}\cdot\vec{\sigma}} & 0 \\ 0 & e^{-\frac{i}{2}\alpha \vec{n}\cdot\vec{\sigma}} \end{pmatrix} \psi \qquad \psi \longrightarrow \begin{pmatrix} e^{-\frac{1}{2}\vec{\chi}\cdot\vec{\sigma}} & 0 \\ 0 & e^{\frac{1}{2}\vec{\chi}\vec{\sigma}} \end{pmatrix} \psi.$$

Thus, the subspaces spanned by $\psi_-^T = (\psi^1, \psi^2, 0, 0)$ and $\psi_+^T = (0, 0, \psi^3, \psi^4)$ transform separately.

Irrespective of the concrete representation, the reducibility of the Dirac spinor representation as a representation for Spin(1,3) can be seen as follows: Define $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, then the orthogonal projection oper.

$$P_\pm = \frac{1}{2}(\mathbb{1}_4 \pm \gamma^5)$$

have the properties $(P_\pm)^2 = P_\pm$, $P_+ P_- = 0 = P_- P_+$, and $P_+ + P_- = \mathbb{1}$

If we now define $\Psi_\pm := P_\pm \Psi$, we have $P_\pm \Psi_\mp = 0$. Since $[S^{\rho\sigma}, \gamma^5] = 0$, we have $P_\mp S[\Lambda]\Psi_\pm = 0$, i.e. the + and − subspaces of the Cliff(1,3) algebra transform separately under Spin(1,3).
Remark: The $\Psi_\pm$ are called positive (negative) chirality spinors.

c) Consider a scalar field $\phi(x)$ in D (instead of 4) dimensions with action

$$S = \int d^D x \left(\frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 \right) \tag{6}$$

State the mass dimension of $\phi(x)$.
Hint: In units where $\hbar = c = 1$, what is the mass dimension of an action?
Irrespective of the number of spacetime dimensions D, the action S in natural units has mass dimension $[S] = 0$. Now since both space and time have mass dimension $[x^\mu] = -1 \; \forall \mu$, every term in the Lagrangian needs $[\mathcal{L}] = D$ for $[S]$ to be zero.
Together with $[m^2] = 2$, this requires

$$D = [\mathcal{L}] = [m^2] + [\phi] = 2 + [\phi^2] \implies [\phi] = \frac{1}{2}(D-2) = \frac{D}{2} - 1$$

d) Consider the process $e^+ + e^- \to e^+ + e^-$ in Quantum Electrodynamics. Argue that this process is ultra-violet finite to every order in perturbation theory.
Hint: Draw the process in terms of resummed 1PI subdiagrams – no explicit expansion order by order in perturbation theory. Use general properties and relations of these resummed 1PI diagrams to show that all divergencies cancel!
Not discussed in our treatment of QFT! Will follow at the beginning of QFT II.