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General Relativity – Exercise Sheet 9

Problem 1 (Coordinate change) [10 points]

Given the line element

normally $d\Omega$

$$ds^2 = A(r)\, c^2 dt^2 - B(r)\, dr^2 - C(r)\, r^2 d\Omega^2,$$

define $B(r)$ to be $B(r) = G(r) + C(r)$. Transform the radial coordinate according to

$$\frac{d\rho}{\rho} = \frac{dr}{r}\sqrt{1 + \frac{G(r)}{C(r)}},$$

and show that the line element can be brought to the form

$$ds^2 = X(\rho)\, c^2 dt^2 - Y(\rho)\left(d\rho^2 + \rho^2 d\Omega^2\right),$$

where $X(\rho) = A(r)$, $Y(\rho) = \frac{r^2}{\rho^2} C(r)$, thereby uncovering a spatially isotropic form of the metric.

$$ds^2 = A(r)\, c^2 dt^2 - \underbrace{B(r)}_{G(r)+C(r)} dr^2 - C(r)\, r^2 d\Omega^2 = A(r)\, c^2 dt^2 - C(r)\, r^2 \Big(\underbrace{\Big(1 + \frac{G(r)}{C(r)}\Big)\frac{dr^2}{r^2}}_{d\rho^2/\rho^2} + d\Omega^2\Big)$$

$$= \underbrace{A(r)}_{X(\rho)}\, c^2 dt^2 - \underbrace{C(r)\frac{r^2}{\rho^2}}_{Y(\rho)}\left(d\rho^2 + \rho^2 d\Omega^2\right) = X(\rho)\, c^2 dt^2 - Y(\rho)\left(d\rho^2 + \rho^2 d\Omega^2\right)$$

Problem 2 (Black hole evasion) [15 points]

A space ship is travelling at $\dot{r}_\infty = \frac{c}{\sqrt{2}}$ (freely falling), when the crew notice that they are going to pass within impact parameter $b = 4R_s$ of a black hole. (Assume their journey starts at $r = \infty$.)

The spaceship's trajectory is governed by the equation of motion derived on the last sheet,

$$\frac{1}{2}\dot{r}^2 + V_{\text{eff}}(r) = \alpha, \qquad \text{with } \alpha \in \mathbb{R} \text{ constant},$$

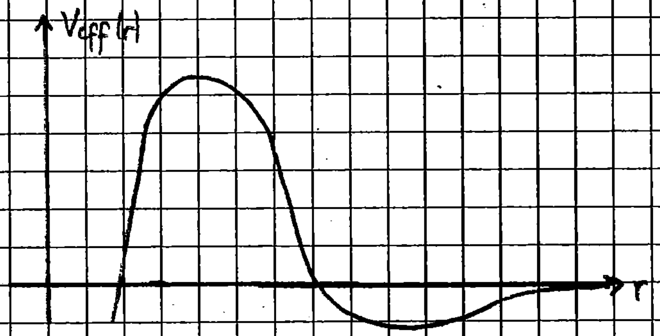
with the effective potential

$$V_{eff}(r) = -\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{c^2r^3}.$$

The spaceships angular momentum is given by $\dot{r}_\infty b$.

Will the spaceship be captured or escape the black hole's pull?

As shown in problem 2 of the last assignment, $V_{eff}(r)$'s plot is of the form



So the question can be answered by checking if the spaceship's energy is large enough to traverse the black hole's potential barrier.

Since the energy is constant, we can calculate it for all r by looking at $E(r,\dot{r}) = \frac{\dot{r}^2}{2} + V_{eff}(r)$ for $r = r_\infty = \infty$ and $\dot{r} = \dot{r}_\infty = \frac{c}{\sqrt{2}}$.

Since $V_{eff}(r = r_\infty) = 0$, we simply have

$$E(r = r_\infty, \dot{r} = \dot{r}_\infty) = \frac{c^2}{4} = E(r,\dot{r}) \quad \forall\, r, \dot{r}.$$

Now we simply need to calculate the potential barrier height to decide on the question.

$$0 \overset{!}{=} \frac{d}{dr} V_{eff}(r) = \frac{GM}{r^2} - \frac{L^2}{r^3} + \frac{3GML^2}{c^2r^4}$$

Since we are not interested in $V_{eff}(r=0)$, we may assume r to be unequal to zero and multiply the above equation by r^4 to get

$$r^2 - \frac{L^2}{GM} r + \frac{3L^2}{c^2} = \left(r - \frac{L^2}{2GM}\right)^2 - \frac{L^4}{4G^2M^2} + \frac{3L^2}{c^2} = 0$$

Solving for r and inserting what we know about L, i.e.

$L = \dot{r}_\infty b = \frac{c}{\sqrt{2}} 4R_s = \sqrt{8}\, c R_s$, we get the following positions for potential extrema of $V_{eff}(r)$,

$$r_\pm = \frac{l^2}{2GM} \pm \sqrt{\frac{l^4}{4G^2M^2} - \frac{3l^2}{c^2}} = \frac{8c^2R_s^2}{2GM} \pm \sqrt{\frac{64\,c^4R_s^4}{4G^2M^2} - \frac{24c^2R_s^2}{c^2}}$$

$$= 8R_s \pm \sqrt{64R_s^2 - 24R_s^2} = (8 \pm \sqrt{40})\,R_s =: \gamma_\pm R_s$$

Without checking, we assume the smaller value, $\gamma_- R_s$, to be the position of the maximum, i.e. the potential barrier. Then we get

$$V_{eff}(r=r_-) = -\frac{GM}{\gamma_- R_s} + \frac{l^2}{2\gamma_-^2 R_s^2}\left(1 - \frac{2GM}{c^2\gamma_- R_s}\right)$$

$$= -\frac{c^2}{2\gamma_-} + \frac{4c^2}{\gamma_-^2} \approx 0.276\,c^2 > \frac{c^2}{4} = E(r,\dot{r})$$

Therefore, the spaceship rebounds before reaching the top of the potential barrier.

<u>Problem 3 (Gravitational field of a rotating sphere)</u> [15 points]

The linearized field equations can be written as

$$\partial^\sigma \partial_\sigma h_{\mu\nu} = -\frac{16\pi G}{c^4}\left(T_{\mu\nu} - \frac{T}{2}\eta_{\mu\nu}\right).$$

The solution for $h_{\mu\nu}$, known from electrodynamics, is

$$h_{\mu\nu}(\vec{r},t) = -\frac{4G}{c^4}\int d^3r' \,\frac{U_{\mu\nu}(\vec{r}', t - |\vec{r} - \vec{r}'|/c)}{|\vec{r} - \vec{r}'|},$$

where $U_{\mu\nu} = T_{\mu\nu} - \frac{T}{2}\eta_{\mu\nu}$. The energy-momentum tensor reads

$$T^{\mu\nu} = \left(\rho + \frac{P}{c^2}\right)u^\mu u^\nu - P\eta^{\mu\nu}.$$

Assume the source of the gravitational field to be sphere with radius R and homogeneous mass density ρ, rotating with angular frequency ω. Assume $P = 0$ and only keep terms linear in $\frac{v}{c} = \frac{\omega R}{c}$. Calculate the static fields $h_{\mu\nu}(\vec{r})$.

In the static case of a time-independent gravitational field, the time derivatives vanish and the linearized field equations become

$$\partial^\sigma \partial_\sigma h_{\mu\nu} = \Delta h_{\mu\nu} - \underbrace{\partial_{ct}^2 h_{\mu\nu}}_{0} = \Delta h_{\mu\nu} = -\frac{16\pi G}{c^4}\left(T_{\mu\nu} - \frac{T}{2}\eta_{\mu\nu}\right),$$

which is just Poisson's equation, solved by

$$h_{\mu\nu}(\vec{r}) = -\frac{4G}{c^4}\int d^3r' \frac{U_{\mu\nu}(\vec{r}')}{|\vec{r}-\vec{r}'|}. \checkmark$$

Now we turn our attention towards $U_{\mu\nu}(\vec{r})$. Applying approximations as indicated by the exercise ($P=0$, $\frac{v^2}{c^2} = \frac{\omega^2 \rho^2}{c^2} \approx 0$) we can specify $\overset{\frac{v^2}{c^2}\to 0}{\downarrow}$ not $\mathcal{O}(v^2/c^2)$

$$T_{\mu\nu} = \left(\rho + \frac{P}{c^2}\right)u_\mu u_\nu - P\eta_{\mu\nu} \overset{P=0}{=} \rho u_\mu u_\nu, \quad \text{with} \quad u_\mu = \gamma\begin{pmatrix}c\\ \vec{v}\end{pmatrix} \overset{\downarrow}{=} \begin{pmatrix}c\\ \vec{v}\end{pmatrix}$$

$$T = T_\mu{}^\mu = \rho u_\mu u^\mu = \rho(c^2 - \vec{v}^2) = c^2\rho\Big(1 - \underbrace{\frac{v^2}{c^2}}_{0}\Big) = c^2\rho$$

$$U_{\mu\nu} = T_{\mu\nu} - \frac{T}{2}\eta_{\mu\nu} = \rho u_\mu u_\nu - \frac{c^2}{2}\rho\,\eta_{\mu\nu} = \rho\left(u_\mu u_\nu - \frac{c^2}{2}\eta_{\mu\nu}\right)$$

We can calculate each of the ten static fields $h_{\mu\nu}(\vec{r})$ by looking at every component of $U_{\mu\nu}$ separately

$$U_{00} = \rho\left(u_0 u_0 - \frac{c^2}{2}\eta_{00}\right) = \rho\left(c^2 - \frac{c^2}{2}\right) = \frac{c^2}{2}\rho$$

$$U_{11} = \rho\left(u_1 u_1 - \frac{c^2}{2}\eta_{11}\right) = \rho\left(v_x^2 + \frac{c^2}{2}\right) = \frac{c^2}{2}\rho\Big(1 + \underbrace{\frac{v_x^2}{c^2}}_{0}\Big) = \frac{c^2}{2}\rho = U_{22} = U_{33}$$

$$U_{01} = \rho\left(u_0 u_1 - \frac{c^2}{2}\eta_{01}\right) = c\rho v_x, \quad U_{02} = c\rho v_y, \quad U_{03} = c\rho v_z$$

$$U_{12} = \rho\left(u_1 u_2 - \frac{c^2}{2}\eta_{12}\right) = \rho v_x v_y, \quad U_{13} = \rho v_x v_z, \quad U_{23} = \rho v_y v_z.$$

$U_{\mu\nu}$'s other six entries are determined by its symmetry under exchange of μ and ν (which $U_{\mu\nu}$ inherited from $T_{\mu\nu}$ and $\eta_{\mu\nu}$).

$$\underline{U} = \rho c^2 \begin{pmatrix} \frac{1}{2} & v_x/c & v_y/c & v_z/c \\ v_x/c & \frac{1}{2} & v_x v_y/c^2 & v_x v_z/c^2 \\ v_y/c & v_y v_x/c^2 & \frac{1}{2} & v_y v_z/c^2 \\ v_z/c & v_z v_x/c^2 & v_z v_y/c^2 & \frac{1}{2} \end{pmatrix} \overset{\frac{v^2}{c^2}=0}{\downarrow}{=} \rho c^2 \left(\begin{array}{c|ccc} 1/2 & & \vec{v}^T/c & \\ \hline \vec{v}/c & 1/2 & & \\ & & 1/2 & \\ & & & 1/2 \end{array}\right) \checkmark$$

To bring out the $\vec{r}$-dependence of $\underline{U} = \underline{U}(\vec{r})$, we note that any vector $\vec{r}$ that rotates with angular velocity $\omega = |\vec{\omega}|$ around an axis with angular speed vector $\vec{\omega}$ satisfies

$$\vec{v} = \vec{\omega} \times \vec{r}.$$

By choosing spherical coordinates such that $\vec{r}\vec{r}' = r r' \cos\vartheta$ and such that the origin, $\vec{r} = 0$, coincides with the sphere's center, and reminding ourselves

$$\rho(\vec{r}) = \begin{cases} \rho, & \text{if } \vec{r} \in \partial B_R(\vec{r}=0) = \{\vec{r} \in \mathbb{R}^3 \mid |\vec{r}| = R\} \\ 0, & \text{otherwise}, \end{cases}$$

we obtain the following static gravitational fields

ρ is distributed not only along surface

$$h_{ii}(\vec{r}) = -\frac{4G}{c^4}\int_{\mathbb{R}^3} d^3r' \frac{U_{ii}(\vec{r}')}{|\vec{r}-\vec{r}'|} = -\frac{2G}{c^2}\int_{\mathbb{R}^3} d^3r' \frac{\rho(\vec{r}')}{|\vec{r}-\vec{r}'|} = -\frac{2G}{c^2}\int_0^\infty dr' \int_0^\pi d\vartheta \int_0^{2\pi} d\varphi \frac{\rho\,\delta(r'-R)\, r'^2 \sin\vartheta}{(r^2+r'^2-2rr'\cos\vartheta)^{\frac{1}{2}}}$$

$$= -\frac{2G\rho}{c^2}\int_0^\pi d\vartheta \int_0^{2\pi} d\varphi \frac{R^2 \sin\vartheta}{(r^2+R^2-2rR\cos\vartheta)^{\frac{1}{2}}}, \quad u = r^2+R^2-2rR\cos\vartheta$$

$$du = 2rR\sin\vartheta$$

$$= -\frac{4\pi G\rho}{c^2}\int_{a:=r^2+R^2-2rR}^{b:=r^2+R^2+2rR} \frac{du}{2rR}\frac{R^2}{\sqrt{u}} = -\frac{4\pi G\rho}{c^2}\frac{R}{r}\Big(\underbrace{\sqrt{r^2+R^2+2rR}}_{r+R} - \underbrace{\sqrt{r^2+R^2-2rR}}_{|r-R|}\Big)$$

$$= -\frac{4\pi G\rho}{c^2}\begin{cases} \frac{R}{r}(r+R-r+R) = \frac{2R^2}{r}, & \text{for } r \geq R \\ \frac{R}{r}(r+R-R+r) = 2R, & \text{for } r < R \end{cases}, \quad \text{where } i \in \{0,1,2,3\}$$

$$\vec{h}(\vec{r}) = \begin{pmatrix} h_{01}(\vec{r}) \\ h_{02}(\vec{r}) \\ h_{03}(\vec{r}) \end{pmatrix} = -\frac{4G}{c^4}\int_{\mathbb{R}^3} d^3r \frac{\vec{U}(\vec{r})}{|\vec{r}-\vec{r}'|} = -\frac{4G\rho}{c^3}\int_{\mathbb{R}^3} d^3r' \frac{\rho(\vec{r}')\,\vec{\omega}\times\vec{r}'}{|\vec{r}-\vec{r}'|}$$

$$= -\frac{4G}{c^3}\int_0^\infty dr' \int_0^\pi d\vartheta \int_0^{2\pi} d\varphi \frac{\rho\,\delta(r'-R)\, r'^2 \sin\vartheta}{(r^2+r'^2-2rr'\cos\vartheta)^{\frac{1}{2}}}\, \vec{\omega} \times r' \begin{pmatrix} \cos\varphi \sin\vartheta \\ \sin\varphi \sin\vartheta \\ \cos\vartheta \end{pmatrix}$$

$$= -\frac{8\pi G\rho}{c^3}\int_0^\pi d\vartheta \frac{R^3 \sin\vartheta}{(r^2+R^2-2rR\cos\vartheta)^{\frac{1}{2}}}\, \vec{\omega} \times \begin{pmatrix} 0 \\ 0 \\ \cos\vartheta \end{pmatrix}, \quad u = r^2+R^2-2rR\cos\vartheta$$

$$du = 2rR\sin\vartheta$$

$$= -\frac{8\pi G\rho}{c^3}\,\vec{\omega}\times\vec{e}_z \int_a^b du \frac{1}{4r^2}\frac{r^2+R^2-u}{\sqrt{u}}, \quad v = \sqrt{u}, \quad dv = \frac{1}{2\sqrt{u}}du$$

$$= -\frac{8\pi G\rho}{c^3}\vec{\omega}\times\vec{e}_z\int_{\sqrt{a}}^{\sqrt{b}} dv\, \frac{1}{2r^2}\left(r^2+R^2-v^2\right)$$

$$= -\frac{4\pi G\rho}{c^3 r^2}\vec{\omega}\times\vec{e}_z\left[\left(r^2+R^2\right)\left(\sqrt{b}-\sqrt{a}\right)-\frac{1}{3}\left(b^{\frac{3}{2}}-a^{\frac{3}{2}}\right)\right]$$

$$= -\frac{4\pi G\rho}{c^3 r^2}\vec{\omega}\times\vec{e}_z\left[\left(r^2+R^2\right)\left(r+R-|r-R|\right)-\frac{1}{3}\left((r+R)^3-|r-R|^3\right)\right]$$

$$= \left\{\begin{array}{ll} -\frac{4\pi G\rho}{c^3 r^2}\vec{\omega}\times\vec{e}_z\,\frac{4}{3}R^3 = -\frac{16\pi G\rho R^3}{3c^3 r^2}\,\vec{\omega}\times\vec{e}_z, & \text{for } r\geq R \\ -\frac{4\pi G\rho}{c^3 r^2}\vec{\omega}\times\vec{e}_z\,\frac{4}{3}r^3 = -\frac{16\pi G\rho r}{3c^3}\,\vec{\omega}\times\vec{e}_z, & \text{for } r< R \end{array}\right\} = \left(h^i(\vec{r})\right)^T$$

Interestingly, due to $\vec{\omega}\times\vec{e}_z$, the off-diagonal elements of $\underline{h}(\vec{r})$ vanish along the axis of rotation ($\vec{\omega}\times\vec{e}_z$ has no component along $\vec{\omega}$), so as one would expect in the case of a perfect sphere, an observer sitting on the axis of rotation cannot (based on the gravitational field) distinguish between a rotating and a non-rotating field source.

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Problem 4 (Extra: Falling into a black hole) [5 points]

How long would a fall into a black hole take for a freely falling observer? What does his more fortunate friend at infinite distance say?

As time-dilation diverges upon approaching a black hole's event horizon, nothing can ever truly pass that horizon and be undetectable outside the black hole for good. However, there is no limit to slowing down of time, so the gaps between individual photons sent out by the object falling into the black hole can quickly increase to become billions of years. | It can be never observed $t\to\infty$

On the other hand, a falling observer, since he is freely falling, would detect nothing out of the ordinary with regard to his proper time.

-2