String Theory

Solution to Assignment 9

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1 Gaussian integral

Our aim is to prove the formula

$$S_{\mathbb{S}^2}(\boldsymbol{k}_1,\dots,\boldsymbol{k}_n) = \Big\langle \prod_{j=1}^{n} \mathcal{N}\big(e^{ik_j\cdot X(z_j,\bar z_j)}\big)\Big\rangle \propto \delta^D\Big(\sum_{j<l}^{n} k_j\Big)\prod_{j<l}|z_j - z_l|^{\alpha' k_j\cdot k_l} \tag{1}$$

for the tree-level S-matrix describing the scattering of n closed string tachyons propagating in D dimensions.
We do so by evaluating the functional integral

$$Z_{\mathbb{S}^2}[J] = \int \mathcal{D}X \exp\left[\int_{\mathbb{C}} \mathrm{d}z\mathrm{d}\bar z \left(\frac{1}{2\pi\alpha'} X\cdot\partial\bar\partial X + iJ\cdot X\right)\right] \tag{2}$$

involving the source $J^\mu(z,\bar z)$.

a) Give the form of $J^\mu(z,\bar z)$ relevant to the S-matrix (1) and convince yourself that indeed the problem reduces to the one defined in eq. (2).

b) This Gaussian integral is best performed by expanding the fields into eigenmodes $X_I(z,\bar z)$ of the Laplace operator $\partial\bar\partial$:

$$\partial\bar\partial X_I(z,\bar z) = -\omega_I^2 X_I(z,\bar z). \tag{3}$$

These form a complete set:

$$X^\mu(z,\bar z) = \sum_I x_I^\mu X_I(z,\bar z), \qquad x_I^\mu = \int_{\mathbb{C}} \mathrm{d}z\mathrm{d}\bar z\, X^\mu(z,\bar z) X_I(z,\bar z), \tag{4}$$

$$\int_{\mathbb{C}} \mathrm{d}z\mathrm{d}\bar z\, X_I(z,\bar z)X_{I'}(z,\bar z) = \delta_{II'}, \qquad \sum_I X_I(z,\bar z)X_I(z',\bar z') = \delta(z-z')\,\delta(\bar z-\bar z'). \tag{5}$$

Use this to arrive at

$$Z_{\mathbb{S}^2}[J] = \prod_{I,\mu}\int \mathrm{d}x_I^\mu\, \exp\left(-\frac{w_I^2}{2\pi\alpha'} x_I^\mu x_{I,\mu} + i x_I^\mu J_{I,\mu}\right), \tag{6}$$

where

$$J_I^\mu = \int_{\mathbb{C}} \mathrm{d}z\mathrm{d}\bar z\, J^\mu(z,\bar z)\, X_I(z,\bar z). \tag{7}$$

c) By considering the zero mode $I = 0$ with $\omega_0 = 0$ and the non-zero modes $I \neq 0$ separately argue that

$$Z_{\mathbb{S}^2}[J] = i(2\pi)^D \delta^D(J_0) \prod_{I\neq 0} \left(\frac{2\pi^2\alpha'}{\omega_I^2}\right)^{\frac{D}{2}} \exp\left(-\frac{\pi\alpha' J_I^\mu J_{I,\mu}}{2\omega_I^2}\right). \tag{8}$$

d) This can be rewritten as

$$Z_{\mathbb{S}^2}[J] = i(2\pi)^D \delta^D(J_0) \det{}'\Big(-\frac{\partial\bar{\partial}}{2\pi^2\alpha'}\Big)^{-\frac{D}{2}} \exp\Big(-\frac{1}{2}\int_{\mathbb{C}} \mathrm{d}z\mathrm{d}\bar{z}\, \mathrm{d}z'\mathrm{d}\bar{z}'\, J(z,\bar{z})\cdot J(z',\bar{z}')\; G'(z,\bar{z},z',\bar{z}')\Big) \tag{9}$$

where the the notation $\det'$ indicates that we omit zero modes in the functional determinant. Argue that the Green's function $G'(z,\bar{z},z',\bar{z}')$ satisfies

$$-\frac{1}{\pi\alpha'}\partial\bar{\partial}\, G'(z,\bar{z},z',\bar{z}') = \sum_{I\neq 0} X_I(z,\bar{z}) X_I(z',\bar{z}') = \delta(z-z')\,\delta(\bar{z}-\bar{z}') - X_0^2. \tag{10}$$

Remark: Without proof we state that the effect of X_0^2 above is that self-contractions of the fields at $z_i = z_j$ are cancelled. With this knowledge we can continue with the Green's function $G(z,\bar{z},z',\bar{z}')$ where the term X_0^2 in eq. (10) is omitted and omit self-contractions. Argue that

$$G(z,\bar{z},z',\bar{z}') = -\frac{\alpha'}{2}\ln(|z-z'|^2). \tag{11}$$

e) Now we return to the amplitude (1). Deduce this result from the above considerations.

a)

> **Note:** Equation (1) is already a simplified expression valid only at tree-level. Normally, to obtain the full scattering amplitude, the S-matrix would contain a sum over all compact world-sheet topologies as well as integrals over the moduli spaces of these topologies, and be divided by the volume of Weyl transformations and diffeomorphisms $\mathrm{Vol}_{\mathrm{Weyl}\times\mathrm{diff}}$. However, since we restrict ourselves to the tree-level S-matrix contribution on the sphere where no such moduli exist, the S-matrix simplifies to the one given in eq. (1).

To generate the scattering matrix (1) from the partition function (2), we need to insert the source

$$J^\mu(z,\bar{z}) = \sum_{j=1}^{n} k_j^\mu\, \delta(z-z_j)\,\delta(\bar{z}-\bar{z}_j), \tag{12}$$

since then

$$\begin{aligned} \frac{Z_{\mathbb{S}^2}[J]}{Z_{\mathbb{S}^2}} &= \frac{1}{Z_{\mathbb{S}^2}} \int \mathcal{D}X \exp\Big[\int_{\mathbb{C}} \mathrm{d}z\mathrm{d}\bar{z}\Big(\frac{1}{2\pi\alpha'} X\cdot\partial\bar{\partial}X + iJ\cdot X\Big)\Big] \\ &= \frac{1}{Z_{\mathbb{S}^2}} \int \mathcal{D}X \exp\Big[\int_{\mathbb{C}} \mathrm{d}z\mathrm{d}\bar{z}\Big(\frac{1}{2\pi\alpha'} X\cdot\partial\bar{\partial}X\Big) + \sum_{j=1}^{n} ik_j\cdot X\Big] \\ &= \frac{1}{Z_{\mathbb{S}^2}} \int \mathcal{D}X e^{-S[X]} \prod_{j=1}^{n} \mathcal{N}\big(e^{ik_j\cdot X}\big) = \Big\langle \prod_{j=1}^{n} \mathcal{N}\big(e^{ik_j\cdot X}\big)\Big\rangle = S_{\mathbb{S}^2}(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n), \end{aligned} \tag{13}$$

where we used that after partial integration (with boundary terms vanishing at infinity), the action of the free boson on the sphere can be written as

$$S[X] = \frac{1}{2\pi\alpha'}\int_{\mathbb{C}} \mathrm{d}z\mathrm{d}\bar{z}\, \partial_z X_\mu(z,\bar{z})\,\partial_{\bar{z}} X^\mu(z,\bar{z}) = -\frac{1}{2\pi\alpha'}\int_{\mathbb{C}} \mathrm{d}z\mathrm{d}\bar{z}\, X_\mu(z,\bar{z})\,\partial_z\partial_{\bar{z}} X^\mu(z,\bar{z}). \tag{14}$$

b) Expanding the exponent in the partition function into eigenmodes $X_I(z, \bar{z})$ of the complex Laplace operator $\partial\bar{\partial}$, we get

$$\begin{aligned}\int_{\mathbb{C}} \mathrm{d}z\mathrm{d}\bar{z}\, X_\mu \,\partial\bar{\partial} X^\mu &\overset{(4)}{=} \int_{\mathbb{C}} \mathrm{d}z\mathrm{d}\bar{z}\, X_\mu \,\partial\bar{\partial}\Big(\sum_I x_I^\mu X_I(z,\bar{z})\Big) \\ &\overset{(3)}{=} -\sum_I \omega_I^2 x_I^\mu \int_{\mathbb{C}} \mathrm{d}z\mathrm{d}\bar{z}\, X_\mu X_I(z,\bar{z}) \overset{(4)}{=} -\sum_I \omega_I^2 x_I^\mu x_{I,\mu}.\end{aligned} \tag{15}$$

Similarly, the second exponential becomes

$$\int_{\mathbb{C}} \mathrm{d}z\mathrm{d}\bar{z}\, J_\mu \, X^\mu \overset{(4)}{=} \int_{\mathbb{C}} \mathrm{d}z\mathrm{d}\bar{z}\, J_\mu \sum_I x_I^\mu X_I(z,\bar{z}) \overset{(7)}{=} \sum_I x_{I,\mu} J_I^\mu. \tag{16}$$

Using that the path integration $\mathcal{D}X$ over all field configurations can be rewritten as the product of integrations over all mode coefficients x_I^μ, i.e.

$$\mathcal{D}X = \prod_\mu \mathcal{D}X^\mu = \prod_{\mu,I} \mathrm{d}x_I^\mu, \tag{17}$$

we can write the partition function as

$$\begin{aligned} Z_{\mathbb{S}^2}[J] &= \int\Big(\prod_{I,\mu} \mathrm{d}x_I^\mu\Big) \exp\Big(-\sum_I \frac{w_I^2}{2\pi\alpha'} x_I^\nu x_{I,\nu} + i\sum_I x_{I,\nu} J_I^\nu\Big) \\ &= \int \prod_{I,\mu} \mathrm{d}x_I^\mu \, \exp\Big(-\frac{w_I^2}{2\pi\alpha'} x_I^\nu x_{I,\nu} + i x_I^\nu J_{I,\nu}\Big).\end{aligned} \tag{18}$$

Since the integrand now factorizes both into dimensions ν and into mode numbers I, we can write the whole multi-dimensional integral as a product of many one-dimensional integrations,

$$Z_{\mathbb{S}^2}[J] = \prod_{I,\mu} \int_{-\infty}^{\infty} \mathrm{d}x_I^\mu \, \exp\Big(-\frac{w_I^2}{2\pi\alpha'} x_I^\mu x_{I,\mu} + i x_I^\mu J_{I,\mu}\Big). \tag{19}$$

c) Pulling out the $I = 0$-term for which $\omega_0 = 0$,

$$\prod_{\mu=0}^{D-1} \int_{-\infty}^{\infty} \mathrm{d}x_0^\mu \, \exp\Big(-\frac{\omega_0^2}{2\pi\alpha'} x_0^\mu x_{0,\mu} + i x_0^\mu J_{0,\mu}\Big) = \prod_{\mu=0}^{D-1} \int_{-\infty}^{\infty} \mathrm{d}x_0^\mu \, e^{i x_0^\mu J_{0,\mu}}, \tag{20}$$

we can treat it separately using the integral representation of the Dirac delta,

$$\int_{-\infty}^{\infty} \mathrm{d}x \, e^{iJx} = 2\pi\, \delta(J), \tag{21}$$

to get

$$\prod_{\mu=0}^{D-1} \int_{-\infty}^{\infty} \mathrm{d}x_0^\mu \, e^{i x_0^\mu J_{0,\mu}} = \prod_{\mu=0}^{D-1} 2\pi\, \delta(J_0^\mu) = (2\pi)^D\, \delta^D(J_0). \tag{22}$$

As for the remaining $I \neq 0$, we would like to solve them as Gaussian integrals. Note however that since spacetime in string theory is simply flat Minkwoski space,[1] and since μ is a spacetime, not a worldsheet index, the x_I^0-term in $x_I^2 = x_{I,\mu}\, x_I^\mu$ has the wrong sign. This can be remedied by a simple Wick rotation $x_I^0 \to i x_I^0$. Then all scalar products become Euclidean and the integration

[1] In particular, the spacetime metric is completely unaffected by any fancy geometry the string CFT may adopt.

measure $\mathrm{d}x_I^0 \to i\mathrm{d}x_I^0$ produces an extra factor of i. Thus the partition function becomes

$$\begin{aligned} Z_{\mathbb{S}^2}[J] &= i(2\pi)^D\,\delta^D(J_0)\prod_{I\neq 0}\prod_{\mu=0}^{D-1}\int_{-\infty}^{\infty}\mathrm{d}x_I^\mu\,\exp\Big(-\frac{w_I^2}{2\pi\alpha'}x_I^\mu x_{I,\mu}+ix_I^\mu J_{I,\mu}\Big)\\ &= i(2\pi)^D\delta^D(J_0)\prod_{I\neq 0}\prod_{\mu=0}^{D-1}\sqrt{\frac{\pi}{\omega_I^2/(2\pi\alpha')}}\exp\Big(\frac{(iJ_{I,\mu})^2}{4\omega_I^2/(2\pi\alpha')}\Big) \\ &= i(2\pi)^D\delta^D(J_0)\prod_{I\neq 0}\Big(\frac{2\pi^2\alpha'}{\omega_I^2}\Big)^{\frac{D}{2}}\exp\Big(-\frac{\pi\alpha' J_I^\mu J_{I,\mu}}{2\omega_I^2}\Big), \end{aligned} \tag{23}$$

where we used the formula

$$\int_{-\infty}^{\infty}\mathrm{d}x\,e^{-ax^2+bx+c}=\sqrt{\frac{\pi}{a}}\,e^{\frac{b^2}{4a}+c}. \tag{24}$$

d) To see how eq. (9) comes about, note that the operator $\Big(-\frac{\partial\bar\partial}{2\pi^2\alpha'}\Big)^{-\frac{D}{2}}$ has eigenvalues $\Big(\frac{\omega_I^2}{2\pi^2\alpha'}\Big)^{-\frac{D}{2}}$. Since its determinant is precisely the product of its eigenvalues, we have

$$\prod_{I\neq 0}\Big(\frac{\omega_I^2}{2\pi^2\alpha'}\Big)^{-\frac{D}{2}}=\det{}'\Big(-\frac{\partial\bar\partial}{2\pi^2\alpha'}\Big)^{-\frac{D}{2}}. \tag{25}$$

As for the exponential, we can simply reverse our application of eq. (7) to obtain

$$\begin{aligned} \prod_{I\neq 0}\exp\Big(-\frac{\pi\alpha' J_I^\mu J_{I,\mu}}{2\omega_I^2}\Big) &\overset{(7)}{=}\exp\Big(-\sum_{I\neq 0}\frac{\pi\alpha'}{2\omega_I^2}\int_{\mathbb{C}}\mathrm{d}z\mathrm{d}\bar z\,J^\mu(z,\bar z)X_I(z,\bar z)\int_{\mathbb{C}}\mathrm{d}z'\mathrm{d}\bar z'\,J_\mu(z',\bar z')X_I(z',\bar z')\Big)\\ &\equiv \exp\Big(-\frac{1}{2}\int_{\mathbb{C}}\mathrm{d}z\mathrm{d}\bar z\int_{\mathbb{C}}\mathrm{d}z'\mathrm{d}\bar z'\,J^\mu(z,\bar z)J_\mu(z',\bar z')\,G'(z,\bar z,z',\bar z')\Big), \end{aligned} \tag{26}$$

where we defined

$$G'(z,\bar z,z',\bar z')\equiv\sum_{I\neq 0}\frac{\pi\alpha'}{\omega_I^2}X_I(z,\bar z)X_I(z',\bar z'). \tag{27}$$

Inserting eqs. (25) and (26) into eq. (23), we indeed reproduce eq. (9),

$$Z_{\mathbb{S}^2}[J]=i(2\pi)^D\delta^D(J_0)\det{}'\Big(-\frac{\partial\bar\partial}{2\pi^2\alpha'}\Big)^{-\frac{D}{2}}\exp\Big(-\frac{1}{2}\int_{\mathbb{C}}\mathrm{d}z\mathrm{d}\bar z\,\mathrm{d}z'\mathrm{d}\bar z'\,J(z,\bar z)\cdot J(z',\bar z')\;G'(z,\bar z,z',\bar z')\Big) \tag{28}$$

Using eq. (3), i.e. $\partial\bar\partial X_I(z,\bar z)=-\omega_I^2X_I(z,\bar z)$, the definition (27) immediately yields eq. (10),

$$\begin{aligned} -\frac{1}{\pi\alpha'}\partial\bar\partial\,G'(z,\bar z,z',\bar z') &= -\sum_{I\neq 0}\frac{1}{\omega_I^2}\,\partial\bar\partial\,X_I(z,\bar z)X_I(z',\bar z')\\ &= \sum_{I\neq 0}X_I(z,\bar z)X_I(z',\bar z')\overset{(5)}{=}\delta(z-z')\,\delta(\bar z-\bar z')-X_0^2, \end{aligned} \tag{29}$$

where the last equality holds to the eigenmodes' completeness relation.

To demonstrate that the *full* Green's function is given by

$$G(z,\bar z,z',\bar z')=-\frac{\alpha'}{2}\ln(|z-z'|^2), \tag{30}$$

we show that it satisfies eq. (10) with the self-contraction cancelling X_0^2-term omitted,

$$-\frac{1}{\pi\alpha'}\partial\bar\partial\,G'(z,\bar z,z',\bar z')=\frac{1}{2\pi}\partial\bar\partial\ln[(z-z')(\bar z-\bar z')]=\frac{1}{2\pi}\partial\,\frac{(z-z')}{(z-z')(\bar z-\bar z')}=\frac{1}{2\pi}\partial\,\frac{1}{\bar z-\bar z'}. \tag{31}$$

Using now the relation $2\pi\delta(z)\,\delta(\bar z)=\partial_z\frac{1}{\bar z}$ which we proved in exercise 1.b) on assignment 8, we indeed get

$$-\frac{1}{\pi\alpha'}\partial\bar\partial\,G(z,\bar z,z',\bar z')=\delta(z-z')\,\delta(\bar z-\bar z'). \tag{32}$$

> **Note:** Of course, we could add any harmonic function $f(z,\bar z,z',\bar z')$ to $G(z,\bar z,z',\bar z')$ that fulfills $\partial\bar\partial f(z,\bar z,z',\bar z') = 0$ and would still reproduce eq. (10).

e) Coming back to the tree-level scattering matrix of n closed string tachyons propagating in D dimensions, we deduce that it is indeed of the form given in eq. (1) by inserting the source $J^\mu(z,\bar z) = \sum_{j=1}^n k_j^\mu\,\delta(z-z_j)\,\delta(\bar z-\bar z_j)$ as defined in eq. (12) of part a) into the partition function on the sphere as derived in eq. (28), more specifically its exponential part (26),

$$\exp\Big(-\frac{1}{2}\int_{\mathbb{C}}\mathrm{d}z\mathrm{d}\bar z\int_{\mathbb{C}}\mathrm{d}z'\mathrm{d}\bar z'\,J(z,\bar z)\cdot J(z',\bar z')\,G'(z,\bar z,z',\bar z')\Big) = \exp\Big(-\frac{1}{2}\sum_{j\neq l=1}^{n}k_j\cdot k_l\,G(z_j,\bar z_j,z_l',\bar z_l')\Big). \tag{33}$$

Since setting $j\neq l$ takes care of avoiding self-contractions, we may insert the full Green's function (30) to simplify further

$$\exp\Big(-\frac{1}{2}\sum_{j\neq l=1}^{n}k_j\cdot k_l\,G(z_j,\bar z_j,z_l',\bar z_l')\Big)\overset{(30)}{=}\exp\Big(\frac{\alpha'}{4}\sum_{j\neq l=1}^{n}k_j\cdot k_l\,\ln(|z_j-z_l'|^2)\Big). \tag{34}$$

Now it is clear that the sum is symmetric under $j\leftrightarrow l$, so we may restrict to $j<l$ and receive a factor of 2 in return.

$$\exp\Big(\frac{\alpha'}{2}\sum_{j<l}^{n}k_j\cdot k_l\,\ln(|z_j-z_l'|^2)\Big) = \prod_{j<l}^{n}\exp\Big(\ln(|z_j-z_l'|^{\alpha' k_j\cdot k_l})\Big) = \prod_{j<l}^{n}|z_j-z_l'|^{\alpha' k_j\cdot k_l}. \tag{35}$$

Inserting nice and compact expression into the partition function eq. (28), we finally arrive at the tree-level closed string scattering matrix on the sphere,

$$\boxed{\begin{aligned} S_{\mathbb{S}^2}(\boldsymbol{k}_1,\dots,\boldsymbol{k}_n)\overset{(13)}{=}\frac{Z_{\mathbb{S}^2}[J]}{Z_{\mathbb{S}^2}} &= \frac{i(2\pi)^D}{Z_{\mathbb{S}^2}}\delta^D(J_0)\,\det{}'\Big(-\frac{\partial\bar\partial}{2\pi^2\alpha'}\Big)^{-\frac{D}{2}}\prod_{j<l}^{n}|z_j-z_l'|^{\alpha' k_j\cdot k_l} \\ &\propto \delta^D\Big(\sum_{j<l}^{n}k_j\Big)\prod_{j<l}^{n}|z_j-z_l'|^{\alpha' k_j\cdot k_l}, \end{aligned}} \tag{36}$$

where we used that the functional determinant $\det'\big(-\frac{\partial\bar\partial}{2\pi^2\alpha'}\big)^{-\frac{D}{2}}$ is a constant independent of both the k_j and z_j, and that

$$\delta^D(J_0)\overset{(7)}{=}\delta^D\Big(\int_{\mathbb{C}}\mathrm{d}z\mathrm{d}\bar z\,J(z,\bar z)\,X_0(z,\bar z)\Big)\overset{(12)}{=}\delta^D\Big(\sum_{j=1}^{n}k_j\,X_0(z_j,\bar z_j)\Big)\propto\delta^D\Big(\sum_{j=1}^{n}k_j\Big). \tag{37}$$

In the last step, we made use of the fact that the string field $X(z,\bar z)$ lives on the compact space $\mathbb{S}^2$ for which $X_0(z,\bar z) = X_0\in\mathbb{C}$ constant is a valid and, in particular, normalizable choice.

2 The Veneziano Amplitude

The computation of the Veneziano amplitude, i.e. the 4-point amplitude for 4 open string tachyons on the disk, proceeds in a manner analogous to the closed string counterpart discussed in the lecture. The only difference is that the insertion of the vertex operators on the boundary of the disk requires summing over orderings. We perform the computation on the upper half-plane with vertex operators inserted on the real line with coordinate y.

a) Show that

$$S_{\mathbb{D}^2}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4) = ig_o^4\, C^X_{\mathbb{D}^2}(2\pi)^{26}\delta^{(26)}\Big(\sum_i \boldsymbol{k}_1\Big)\int_{-\infty}^{\infty}\mathrm{d}y_4\,|y_4|^{-\alpha' u-2}\,|1-y_4|^{-\alpha' t-2} + (t\leftrightarrow s), \tag{38}$$

after fixing $y_1 = 0$, $y_2 = 1$, $y_3 \to \infty$ by $PSL(2,\mathbb{R})$ invariance.

You may use that

$$\Big\langle \prod_{j=1}^{m} e^{i\boldsymbol{k}_j\cdot \boldsymbol{X}(y_j)}\Big\rangle_{\mathbb{D}^2} = iC^X_{\mathbb{D}^2}(2\pi)^{26}\delta^{(26)}\Big(\sum_{j=1}^{m}\boldsymbol{k}_j\Big)\prod_{j<l}^{m}|y_j-y_l|^{2\alpha' \boldsymbol{k}_j\cdot\boldsymbol{k}_l}. \tag{39}$$

b) Now split the integral into the three ranges $-\infty < y_4 < 0$, $0 < y_4 < 1$, $1 < y_4 < \infty$. Draw the orderings of the vertex operators 1, 2, 3, 4 corresponding to these ranges for both terms in eq. (38). Argue that these ranges can be transformed into each other by $PSL(2,\mathbb{R})$ invariance and deduce

$$S_{\mathbb{S}^2}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4) = 2iC^X_{\mathbb{D}^2}(2\pi)^{26}\delta^{(26)}\Big(\sum_{j=1}^{m}\boldsymbol{k}_1\Big)\Big(I(s,t)+I(t,u)+I(u,s)\Big) \tag{40}$$

with

$$I(s,t) = \int_0^1 \mathrm{d}y\, y^{-\alpha' s-2}(1-y)^{-\alpha' t-2} \equiv B(-\alpha' s-1, -\alpha' t-1). \tag{41}$$

Note: The function $B(a,b)$ is the Euler Beta-function and can be shown to be related to the Euler Γ-function via

$$B(a,b) = \frac{\Gamma(a)\,\Gamma(b)}{\Gamma(a+b)}. \tag{42}$$

a) Before we dive into the calculation, let's collect some background information.

> **Note:** In string theory, scattering processes at tree-level occur exclusively on worldsheets that are Riemann surfaces with positive Euler characteristic χ.[2] Exactly three such surfaces exist:
>
> - The **two-sphere** $\mathbb{S}^2$ acts as a stage for tree-level oriented closed string scattering.
> - The **disk** $\mathbb{D}^2$ hosts oriented open string scatterings. Vertex operators have to be inserted on the boundary.
> - The **real projective plane** $\mathbb{R}P^1$ accommodates scattering processes in unoriented string theory.
>
> Crucially, all these surfaces are moduli-free. As a result, no anti-ghost insertions $\frac{1}{4\pi}(b|\partial_\alpha \hat{h})$ nor integrations $\int_F \mathrm{d}t^\alpha$ over the moduli's fundamental domains are necessary when computing tree-level scatterings in string theory.

We start our calculation from the heuristic expression for the S-matrix given in the lecture:

$$S(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_n) = \sum_{\substack{\text{compact}\\ \text{topologies}}} \frac{\int\mathcal{D}X\int\mathcal{D}h}{\mathrm{Vol}_{\mathrm{Diff}\times\mathrm{Weyl}}} e^{-S_{\mathrm{P}}-\lambda\chi}\prod_{i=1}^{n} V_{\mathrm{ot}}(\boldsymbol{k}_i). \tag{43}$$

The Veneziano amplitude describes scattering of $n = 4$ open string tachyons and incorporates just the tree-level contribution to this process so we truncate the sum over compact topologies right

[2] Of course, for higher-loop corrections more intricate topologies with negative χ arise. The worldsheet topology does, however, *not* depend on the number of states participating in a scattering. This is taken care of entirely by the insertion of vertex operators. Whether we want to compute the vacuum amplitude or two-to-two open string scattering at tree-level, both scenarios take place on the disk, just with four instead of zero open string vertex operators inserted.

after the disk. Note that open string vertex operators are inserted on the worldsheet's boundary. Since different points on the boundary correspond to different τ, the disk admits a notion of time-ordering for the vertex operator insertions. The sum over topologies, even if only carried out to tree-level, contains all six cyclically inequivalent orderings that are possible with $n = 4$ insertions. Equation (43) thus becomes

$$S_{\mathbb{D}^2}(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_4) = \sum_{\pi \in S_4/\mathbb{Z}_4} \frac{\int \mathcal{D}X \int \mathcal{D}h}{\mathrm{Vol}_{\mathrm{Diff}\times\mathrm{Weyl}}} e^{-S_{\mathrm{P}} - \lambda\,\chi_{\mathbb{D}^2}} \prod_{i=1}^4 V_{\mathrm{ot}}(\boldsymbol{k}_i). \tag{44}$$

One of the important the results from our analysis of conformal field theory was the **operator-state correspondence**. In particular, we found that the localized vertex operator $V_{\mathrm{ot}}(y)$ for the creation of one open string tachyon from the $PSL(2,\mathbb{C})$-invariant vacuum is the primary field $\mathcal{N}(e^{i\boldsymbol{k}\cdot\boldsymbol{X}(y)})$. However, what furnishes the integrand of the path integral in eq. (44) is not the localized but the integrated vertex operator. Integration $\int_{\partial\Sigma} \mathrm{d}y$ over all possible insertion points on the worldsheet boundary[3] $\partial\Sigma$ ensures invariance of the full amplitude under worldsheet-diffeomorphisms. Hence

$$V_{\mathrm{ot}}(\boldsymbol{k}_i) = g_o \int_{\partial\Sigma} \mathrm{d}y_i\, \mathcal{N}(e^{i\boldsymbol{k}_i\cdot\boldsymbol{X}(y_i)}), \tag{45}$$

where the normalization conventionally includes one power of the string coupling g_o (respectively $g_c = g_o^2$ for closed string vertex operators). Inserting this expression into eq. (44) yields

$$\begin{aligned} S_{\mathbb{D}^2}(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_4) &= \sum_{\pi \in S_4/\mathbb{Z}_4} \frac{\int \mathcal{D}X \int \mathcal{D}h}{\mathrm{Vol}_{\mathrm{Diff}\times\mathrm{Weyl}}} e^{-S_{\mathrm{P}} - \lambda\,\chi_{\mathbb{D}^2}} \prod_{i=1}^4 g_o \int_{I_\pi} \mathrm{d}y_i\, \mathcal{N}(e^{i\boldsymbol{k}_i\cdot\boldsymbol{X}(y_i)}) \\ &= \frac{g_o^4}{\mathrm{Vol}_{\mathrm{Diff}\times\mathrm{Weyl}}} \sum_{\pi \in S_4/\mathbb{Z}_4} \int_{I_\pi} \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}y_3 \mathrm{d}y_4 \int \mathcal{D}X \int \mathcal{D}h\, e^{-S_{\mathrm{P}} - \lambda\,\chi_{\mathbb{D}^2}} \prod_{i=1}^4 \mathcal{N}(e^{i\boldsymbol{k}_i\cdot\boldsymbol{X}(y_i)}) \\ &= \frac{g_o^4}{\mathrm{Vol}_{\mathrm{Diff}\times\mathrm{Weyl}}} \sum_{\pi \in S_4/\mathbb{Z}_4} \int_{I_\pi} \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}y_3 \mathrm{d}y_4 \Big\langle \prod_{i=1}^4 \mathcal{N}(e^{i\boldsymbol{k}_i\cdot\boldsymbol{X}(y_i)}) \Big\rangle_{\mathbb{D}^2}, \end{aligned} \tag{46}$$

where in the last step we assumed a normalized partition function $Z_{\mathbb{D}^2} = 1$. Each permutation of the insertion points gives rise to a different integration scheme $I_\pi = \{y_{\pi(1)} < y_{\pi(2)} < y_{\pi(3)} < y_{\pi(4)}\}$.

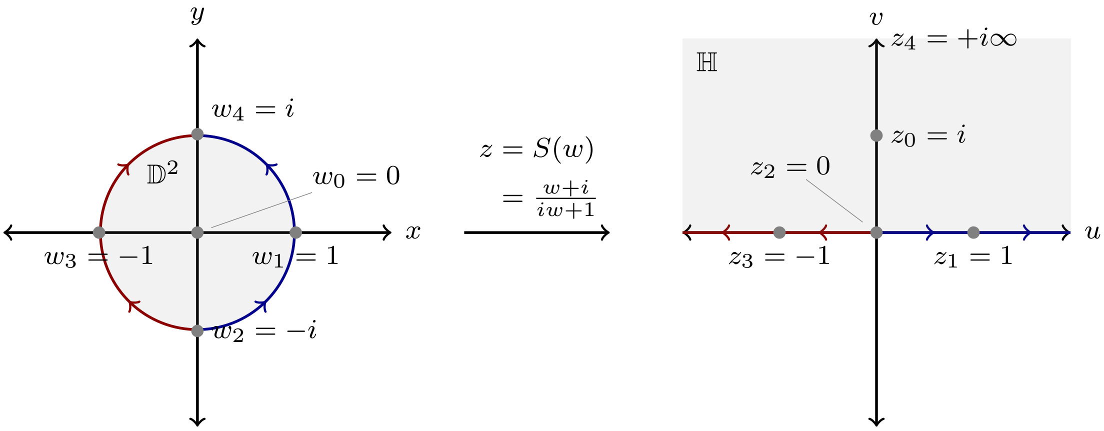
We divide by the volume of the group of diffeomorphisms and Weyl transformations because these are pure gauge degrees of freedom and would lead to overcounting in the path integral. An alternative way to deal with this redundancy is to actually *use* it to fix the position of some of the vertex operator insertions. After all, we only integrated over the insertion points to preserve diffeomorphism invariance on the worldsheet in the first place, so we should be able to reverse this operation until all gauge freedom is used up at which point we are left with an S-matrix containing only truly physical contributions to the scattering amplitude.

The conformal group on $\mathbb{D}^2$ is $PSL(2,\mathbb{R})$, i.e. the Möbius group over $\mathbb{R}$ (hence $\mathrm{Vol}_{\mathrm{Diff}\times\mathrm{Weyl}} = \mathrm{Vol}_{PSL(2,\mathbb{R})}$). We demonstrated in exercise 3 on assignment 7 that $PSL(2,\mathbb{R})$-transformations allow us to map any three distinct points to any other three distinct points on $\mathbb{R}$.

Combined with the conformal map $S(w) = \frac{w+i}{iw+1} \in PSL(2,\mathbb{C})$ which maps $\mathbb{D}^2$ to the half plane $\mathbb{H}$ (following convention, we choose the upper half) and in particular the disk's boundary $\partial\mathbb{D}^2$ to the real line $\partial\mathbb{H} = \mathbb{R} \cup \infty$ compactified by adding the point at infinity[4], this allows us to set e.g. $y_1 = 0$, $y_2 = 1$, $y_3 \to \infty$ and change the integration domain of the remaining variable y_4 from $\partial\Sigma$ to $(-\infty, \infty]$. The action of $S(w)$ is illustrated in the following diagram.

[3] For the closed string, we would integrate over the *whole* worldsheet, i.e. $\int_\Sigma \mathrm{d}z\mathrm{d}\bar{z}$.

[4] Note that the point at $w_4 = i$ is special in the sense that $S(w = i) = i\infty$ as in the illustration, but the counterclockwise limit gives $\lim_{w\to i^+} S(w) = \infty$, whereas the clockwise limit is $\lim_{w\to i^-} S(w) = -\infty$. So the point $w_4 = i$ on the disk corresponds to the compactification point $z = \infty$ where the real line jumps back to $-\infty$.



There is a subtlety that has to be taken care of when fixing the gauge: Following the prescription ensuing eq. (5.45) of the lecture notes for the general gauge-fixed S-matrix, we need to add a c-ghost insertion for every tachyon vertex operator whose position we fix. Constraining $V_{\text{ot}}(\boldsymbol{k}_1)$, $V_{\text{ot}}(\boldsymbol{k}_2)$, and $V_{\text{ot}}(\boldsymbol{k}_3)$, we thus get a factor $\langle c(y_1)\, c(y_2)\, c(y_3)\rangle_{\mathbb{D}^2}$. Invariance of the vacuum under conformal transformations requires that a general CFT n-point function transforms as

$$\langle \phi_i(z_1)\, \phi_j(z_2)\, \phi_k(z_3)\rangle \overset{!}{=} \langle \phi_i'(z_1')\, \phi_j'(z_2')\, \phi_k'(z_3')\rangle. \tag{47}$$

This completely fixes the spacetime-dependence of the three-point function of quasi-primaries to

$$\langle \phi_i(z_1)\, \phi_j(z_2)\, \phi_k(z_3)\rangle \overset{!}{=} \frac{C_{ijk}}{(z_1-z_2)^{h_i+h_j-h_k}\,(z_2-z_3)^{h_j+h_k-h_i}\,(z_1-z_3)^{h_i+h_k-h_j}}, \tag{48}$$

where the structure constant C_{ijk} is closely related to the coefficients appearing in the OPE of the corresponding fields. Since the c-ghosts are primaries of weight $h_c = -1$, eq. (48) for $\langle c(y_1)\, c(y_2)\, c(y_3)\rangle_{\mathbb{D}^2}$ reads

$$\langle c(y_1)\, c(y_2)\, c(y_3)\rangle \propto \frac{1}{(y_1-y_2)^{-1}\,(y_2-y_3)^{-1}\,(y_1-y_3)^{-1}}, \tag{49}$$

Gauge fixing the expectation value in eq. (46) thus reduces it to

$$\begin{aligned}\int_{I_\pi} \mathrm{d}y_1 \mathrm{d}y_2 \mathrm{d}y_3 \mathrm{d}y_4 \left\langle \prod_{i=1}^4 \mathcal{N}(e^{i\boldsymbol{k}_i\cdot\boldsymbol{X}(y_i)})\right\rangle_{\mathbb{D}^2} &= \int_{y_{\pi(1)}}^{y_{\pi(2)}} \mathrm{d}y_4 \left\langle c(y_1)\, c(y_2)\, c(y_3) \prod_{i=1}^4 \mathcal{N}(e^{i\boldsymbol{k}_i\cdot\boldsymbol{X}(y_i)})\right\rangle_{\mathbb{D}^2} \\ &\overset{(49)}{\propto} (y_1-y_2)\,(y_2-y_3)\,(y_1-y_3) \int_{y_{\pi(1)}}^{y_{\pi(2)}} \mathrm{d}y_4 \left\langle \prod_{i=1}^4 \mathcal{N}(e^{i\boldsymbol{k}_i\cdot\boldsymbol{X}(y_i)})\right\rangle_{\mathbb{D}^2},\end{aligned} \tag{50}$$

where in the last we used that the RNS action contains no ghost-boson interaction terms so that the correlators in the two sectors factorize.

For $y_1 = 0$, $y_2 = 1$, $y_3 \to \infty$, the y-terms in front simplify to

$$(y_1-y_2)\,(y_2-y_3)\,(y_1-y_3) = -y_3^2. \tag{51}$$

Dropping the factor $\text{Vol}^{-1}_{\text{Diff}\times\text{Weyl}}$ (since we have now used up all gauge freedom and no longer have to worry about overcounting), and inserting everything y_i-dependent back into eq. (46), we get

$$S_{\mathbb{D}^2}(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4) \propto g_o^4 \sum_{\pi\in S_4/\mathbb{Z}_4} \int_{y_{\pi(1)}}^{y_{\pi(2)}} \mathrm{d}y_4\, y_3^2 \left\langle \prod_{i=1}^4 \mathcal{N}(e^{i\boldsymbol{k}_i\cdot\boldsymbol{X}(y_i)})\right\rangle_{\mathbb{D}^2}. \tag{52}$$

We can now use the hint provided by the exercise and insert eq. (39).

$$S_{\mathbb{D}^2}(\boldsymbol{k}_1,\ldots,\boldsymbol{k}_4) \overset{(39)}{=} i(2\pi)^{26} g_o^4\, C'^X_{\mathbb{D}^2}\, \delta^{(26)}\Big(\sum_{j=1}^m \boldsymbol{k}_j\Big) \sum_{\pi\in S_4/\mathbb{Z}_4} \int_{y_{\pi(1)}}^{y_{\pi(2)}} \mathrm{d}y_4\, y_3^2 \prod_{j<l}^4 |y_j-y_l|^{2\alpha'\boldsymbol{k}_j\cdot\boldsymbol{k}_l}, \tag{53}$$

where we absorbed the proportionality factor from eq. (50) into $C^X_{\mathbb{D}^2} \to C'^X_{\mathbb{D}^2}$, which can be fixed by unitarity later.

For $y_1 = 0$, $y_2 = 1$, $y_3 \to \infty$, the product in eq. (53) simplifies to

$$\begin{aligned} y_3^2 \prod_{j<l}^{4} |y_j - y_l|^{2\alpha' \boldsymbol{k}_j \cdot \boldsymbol{k}_l} &= y_3^2 \underbrace{|y_1 - y_2|}_{1}^{2\alpha' \boldsymbol{k}_1 \cdot \boldsymbol{k}_2} \underbrace{|y_1 - y_3|}_{y_3}^{2\alpha' \boldsymbol{k}_1 \cdot \boldsymbol{k}_3} \underbrace{|y_1 - y_4|}_{|y_4|}^{2\alpha' \boldsymbol{k}_1 \cdot \boldsymbol{k}_4} \\ &\quad \cdot \underbrace{|y_2 - y_3|}_{y_3}^{2\alpha' \boldsymbol{k}_2 \cdot \boldsymbol{k}_3} \underbrace{|y_2 - y_4|}_{|1-y_4|}^{2\alpha' \boldsymbol{k}_2 \cdot \boldsymbol{k}_4} \underbrace{|y_3 - y_4|}_{y_3}^{2\alpha' \boldsymbol{k}_3 \cdot \boldsymbol{k}_4} \\ &= y_3^2 \, y_3^{2\alpha' \overbrace{}^{}\underbrace{(\boldsymbol{k}_1 + \boldsymbol{k}_2 + \boldsymbol{k}_4)}_{-\boldsymbol{k}_3} \cdot \boldsymbol{k}_3} \, |y_4|^{2\alpha' \boldsymbol{k}_1 \cdot \boldsymbol{k}_4} \, |1 - y_4|^{2\alpha' \boldsymbol{k}_2 \cdot \boldsymbol{k}_4} \end{aligned} \tag{54}$$

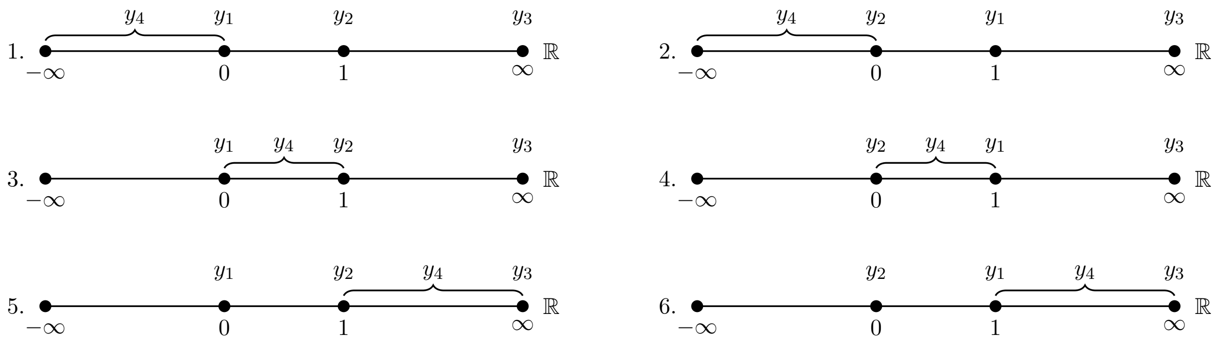
Now we see that sending $y_3 \to \infty$ was chosen wisely, as the y_3-dependence completely disappears,[5]

$$y_3^2 \, y_3^{-2\alpha' \boldsymbol{k}_3^2} = y_3^{2 - 2\alpha' \frac{1}{\alpha'}} = 1. \tag{55}$$

Reinserting the remaining $|y_4|^{2\alpha' \boldsymbol{k}_1 \cdot \boldsymbol{k}_4} \, |1 - y_4|^{2\alpha' \boldsymbol{k}_2 \cdot \boldsymbol{k}_4}$, eq. (53) reads

$$S_{\mathbb{D}^2}(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_4) = i(2\pi)^{26} g_o^4 \, C'^X_{\mathbb{D}^2} \, \delta^{(26)}\Big(\sum_{j=1}^{m} \boldsymbol{k}_j\Big) \sum_{\pi \in S_4/\mathbb{Z}_4} \int_{y_{\pi(1)}}^{y_{\pi(2)}} \mathrm{d}y_4 \, |y_4|^{2\alpha' \boldsymbol{k}_{\pi(1)} \cdot \boldsymbol{k}_4} \, |1 - y_4|^{2\alpha' \boldsymbol{k}_{\pi(2)} \cdot \boldsymbol{k}_4}. \tag{56}$$

It's time to take care of the sum $\sum_{\pi \in S_4/\mathbb{Z}_4}$. Having mapped all insertion points to the real line, the six inequivalent orderings of four vertex operator are as follows:



Note that the order of y_1, y_2, and y_3 is identical on the left and right with only the integrated over y_4 changing place. We can thus add up orderings to get two terms, each with an integration domain of $y_4 \in (-\infty, \infty]$. In the left case we then get $\boldsymbol{k}_{\pi(1)} = \boldsymbol{k}_1$, $\boldsymbol{k}_{\pi(2)} = \boldsymbol{k}_2$, whereas $\boldsymbol{k}_{\pi(1)} = \boldsymbol{k}_1$, $\boldsymbol{k}_{\pi(2)} = \boldsymbol{k}_3$ on the right so that eq. (56) can be written as

$$\begin{aligned} S_{\mathbb{D}^2}(\boldsymbol{k}_1, \ldots, \boldsymbol{k}_4) = i(2\pi)^{26} g_o^4 \, C'^X_{\mathbb{D}^2} \, \delta^{(26)}\Big(\sum_{j=1}^{m} \boldsymbol{k}_j\Big) \int_{-\infty}^{\infty} \mathrm{d}y_4 \, \Big(& |y_4|^{2\alpha' \boldsymbol{k}_1 \cdot \boldsymbol{k}_4} \, |1 - y_4|^{2\alpha' \boldsymbol{k}_2 \cdot \boldsymbol{k}_4} \\ & + |y_4|^{2\alpha' \boldsymbol{k}_1 \cdot \boldsymbol{k}_4} \, |1 - y_4|^{2\alpha' \boldsymbol{k}_3 \cdot \boldsymbol{k}_4} \Big), \end{aligned} \tag{57}$$

Using again the mass-shell condition $\boldsymbol{k}^2 = \frac{1}{\alpha'}$ and the Mandelstam variables $s = -(\boldsymbol{k}_1 + \boldsymbol{k}_2)^2$, $t = -(\boldsymbol{k}_2 + \boldsymbol{k}_4)^2$, $u = -(\boldsymbol{k}_1 + \boldsymbol{k}_4)^2$, we have

$$2\alpha' \boldsymbol{k}_1 \cdot \boldsymbol{k}_4 = \alpha' [(\boldsymbol{k}_1 + \boldsymbol{k}_4)^2 - \boldsymbol{k}_1^2 - \boldsymbol{k}_4^2] = -\alpha' u - 2, \tag{58}$$
$$2\alpha' \boldsymbol{k}_2 \cdot \boldsymbol{k}_4 = \alpha' [(\boldsymbol{k}_2 + \boldsymbol{k}_4)^2 - \boldsymbol{k}_2^2 - \boldsymbol{k}_4^2] = -\alpha' t - 2, \tag{59}$$
$$2\alpha' \boldsymbol{k}_3 \cdot \boldsymbol{k}_4 = \alpha' [(\boldsymbol{k}_3 + \boldsymbol{k}_4)^2 - \boldsymbol{k}_3^2 - \boldsymbol{k}_4^2] = -\alpha' s - 2. \tag{60}$$

[5] As it must since we no longer integrate over y_3 and the final S-matrix should not depend on it.

Reinsertion into eq. (57) yields the S-matrix for the tree-level scattering of four open string tachyons as given in eq. (38),

$$S_{\mathbb{D}^2} = i(2\pi)^{26} g_o^4\, C'^X_{\mathbb{D}^2}\, \delta^{(26)}\Big(\sum_{j=1}^{m} \boldsymbol{k}_j\Big) \int_{-\infty}^{\infty} \mathrm{d}y_4\, |y_4|^{-\alpha' u - 2}\Big(|1-y_4|^{-\alpha' t-2} + |1-y_4|^{-\alpha' s-2}\Big). \tag{61}$$

b) We can also add up the six orderings horizontally in pairs of two rather than vertically in threes. This amounts to splitting the integral in eq. (61) into the three pieces $(-\infty,\infty] \to (-\infty,0] + (0,1] + (1,\infty]$. The middle term then exactly fits eq. (41),

$$I(t,u) + I(t,s) \overset{(41)}{=} \int_0^1 \mathrm{d}y_4\, |y_4|^{-\alpha' t-2}\Big(|1-y_4|^{-\alpha' u-2} + |1-y_4|^{-\alpha' s-2}\Big). \tag{62}$$

Using the $PSL(2,\mathbb{R})$ transformations $x = \frac{1}{-y+1}$ and $x = \frac{1}{y}$ which map $(-\infty,0]$ and $(1,\infty]$ to $(0,1]$, respectively, we can perform a change of variable in the first and third integrals. Thanks to the above mentioned $PSL(2,\mathbb{R})$ invariance, they then each give a contribution equal to eq. (62) up to interchange of vertex operators. Combined, the integrals yield all three cyclic permutations of $I(a,b)$, $a,b \in \{s,t,u\}$ twice so that eq. (61) is equal to

$$S_{\mathbb{S}^2}(\boldsymbol{k}_1,\boldsymbol{k}_2,\boldsymbol{k}_3,\boldsymbol{k}_4) = 2iC^X_{\mathbb{D}^2}(2\pi)^{26}\delta^{(26)}\Big(\sum_{j=1}^{m}\boldsymbol{k}_1\Big)\Big(I(s,t)+I(t,u)+I(u,s)\Big). \tag{63}$$

We get an $I(s,t)$ from orderings 2 and 5, an $I(t,u)$ from orderings 3 and 4, and an $I(u,s)$ from orderings 1 and 6.

3 The moduli space of $\mathbb{T}^2$

Consider a torus $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, parametrized by

$$(\sigma_1,\sigma_2) \mathrel{\hat{=}} (\sigma_1,\sigma_2) + 2\pi(m,n) \quad \text{for} \quad m,n\in\mathbb{Z}. \tag{64}$$

a) Convince yourself that, up to an overall rescaling, the most general metric is given by

$$\mathrm{d}s^2 = |\mathrm{d}\sigma_1 + \tau\mathrm{d}\sigma_2|^2, \qquad \text{for} \quad \tau\in\mathbb{C}. \tag{65}$$

b) Argue that Weyl transformations and diffeomorphisms can be used to define new coordinates ξ_i in terms of which the metric takes the standard form $\mathrm{d}s^2 = |\mathrm{d}\xi_1 + i\mathrm{d}\xi_2|^2$.

Hint: Use that $\chi = \frac{1}{4\pi}\int_{\mathbb{T}^2}\mathrm{d}^2\xi\sqrt{h}R = 2-2g = 0$.

Argue that the new periodicities are given by

$$\xi_1 \simeq \xi_1 + 2\pi(m+nv_1), \qquad \xi_2 \simeq \xi_2 + 2\pi n v_2, \tag{66}$$

for some vector (v_1,v_2).

c) Now define the complex variable $w = \sigma_1 + \tau\sigma_2$ such that in the first picture $\mathrm{d}s^2 = \mathrm{d}w\mathrm{d}\bar{w}$. Convince yourself that the identification (64) implies the identification

$$z \simeq z + 2\pi(m+n\tau), \tag{67}$$

i.e. $\mathbb{T}^2$ is a lattice in $\mathbb{C}$.

Remark: The upshot is the following: We can parametrize the torus in two equivalent ways: In the first picture (64), the coordinates σ_1, σ_2 have standard periodicity 2π, but there appears a modulus τ - the so-called Teichmüller parameter - in the metric (65). Alternatively, we can choose coordinates z, $\bar{z}$ such that the metric has the standard flat form $\mathrm{d}s^2 = \mathrm{d}z\mathrm{d}\bar{z}$, but now in general the periodicity (67) contains a modulus τ that cannot be removed by conformal transformations.

d) Consider now the transformations

$$S : \tau \to \tau + 1, \qquad T : \tau \to -\frac{1}{\tau}, \qquad U : \tau \to \frac{\tau}{\tau + 1}. \tag{68}$$

Determine their action on the parameters (m, n) in eq. (67) and use this to argue that T, S and U leave the torus invariant. Argue that T, S and U generate the group $SL(2, \mathbb{Z})$, whose action on τ is

$$\tau \to \frac{a\tau + b}{c\tau + d}, \qquad \text{where} \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{Z}. \tag{69}$$

Argue that the full modular group of $\mathbb{T}^2$ is $PSL(2, \mathbb{Z}) = SL(2, \mathbb{Z})/\mathbb{Z}_2$.

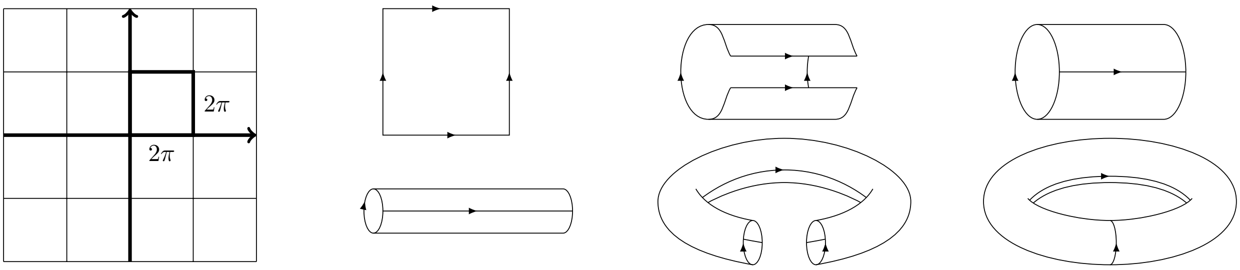
Note: In fact, S and T suffice to generate $SL(2, \mathbb{Z})/\mathbb{Z}_2$, but this is non-trivial to show.

e) Use $PSL(2, \mathbb{Z})$ invariance to show that τ can always be brought to the fundamental domain

$$F_0 = \left\{ z \in \mathbb{C} \,\middle|\, -\frac{1}{2} \le \mathrm{Re}(\tau) \le \frac{1}{2}, \;\; |\tau| \ge 1 \right\}. \tag{70}$$

Represent F_0 graphically and discuss the various identifications of its boundaries.

a) We study the metric on the two-torus $\mathbb{T}^2$ parametrized by the coordinates (σ_1, σ_2) such that σ_1 is identified with $\sigma_1 + 2\pi$ and σ_2 is identified with $\sigma_2 + 2\pi$.[6] The following series of diagrams illustrates how the periodicity $(\sigma_1, \sigma_2) \hat{=} (\sigma_1, \sigma_2) + 2\pi(m, n)$, $m, n \in \mathbb{Z}$ describes a torus.



We start from an arbitrary line element $\mathrm{d}s^2 = h_{ab}\,\mathrm{d}\sigma^a \mathrm{d}\sigma^b$, $a, b \in \{1, 2\}$ with a general worldsheet metric $h_{ab}(\sigma_1, \sigma_2)$. In exercise 2.d) on assignment 2, we had calculated the Ricci scalar's behavior under a Weyl transformation $h_{ab} \to e^{\omega(\sigma_1, \sigma_2)} h_{ab}$ to be

$$\mathcal{R}[e^{\omega} h_{ab}] = e^{-\omega}\,[\mathcal{R} - \nabla^2 \omega]. \tag{71}$$

We can therefore find a suitable Weyl transformation $e^{\omega(\sigma_1, \sigma_2)}$ that makes the Ricci scalar vanish locally. Recalling further our result that due to its many symmetries, the Riemann tensor in two dimensions has just one degree of freedom given by

$$R_{abcd} = \frac{1}{2}(h_{ac} h_{bd} - h_{ad} h_{bc})\mathcal{R}, \tag{72}$$

this automatically implies $R_{abcd} = 0 \;\; \forall\, a, b, c, d \in \{\sigma_1, \sigma_2\}$. Hence in two dimensions, regardless of topology, we can always stretch and bend space such that in a neighborhood around any given point (σ_1, σ_2) on the worldsheet, it is flat. For worldsheets of complicated topology, in particular those with non-zero Euler characteristic χ, there may exist obstructions in the form of moduli that prevent this from working globally. The torus, however, is special in this regard because the global obstruction $\chi = 0$ vanishes. To see this, note that in any dimension d, the Euler characteristic χ

[6] The length of the period is arbitrary. Any value besides 2π could have been chosen. 2π is simplest when parametrizing the embedding into three-dimensional space via sine and cosine.

of a closed (hyper-)surface Σ with $\partial\Sigma = 0$ fulfills the variational identity[7]

$$\delta\chi = \delta\Big(\int_\Sigma \mathrm{d}^d\sigma\,\sqrt{h}\mathcal{R}\Big) = \int_\Sigma \mathrm{d}^d\sigma\,\delta h^{ab}\Big(R_{ab} - \frac{1}{2}\mathcal{R}h_{ab}\Big), \tag{73}$$

where $h = \det(h)$. From this it follows that the Euler characteristic χ is a topological invariant, i.e. remains unchanged under continuous variations of the metric. We can thus reshape the metric into unit form, $h_{ab} = \delta_{ab}$ to explicitly calculate χ on $\mathbb{T}^2$:

$$\chi_{\mathbb{T}^2} = \int_{\mathbb{T}^2} \mathrm{d}^2\sigma\,\sqrt{h}\mathcal{R} \overset{h_{ab}=\delta_{ab}}{\downarrow\!=} \int_0^{2\pi}\int_0^{2\pi} \mathcal{R}\,\mathrm{d}\sigma_1\mathrm{d}\sigma_2 = 0. \tag{74}$$

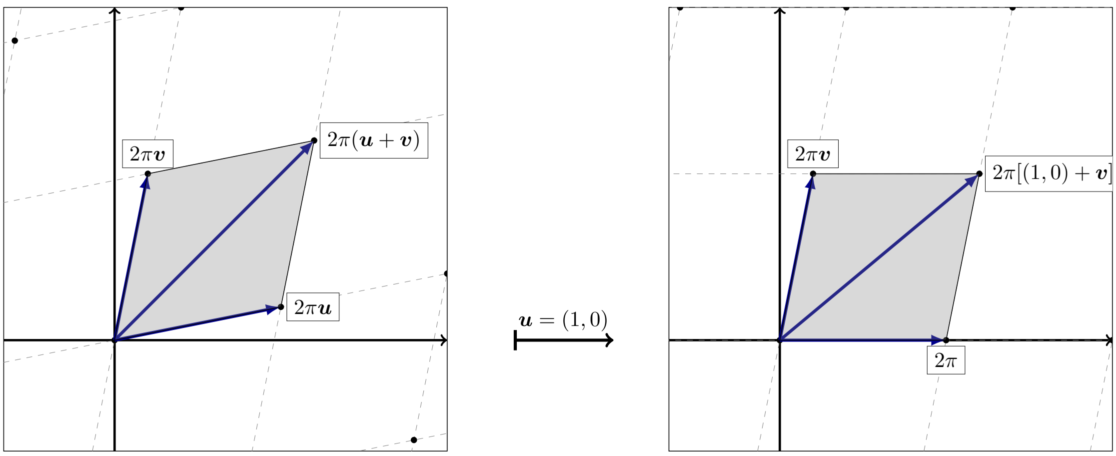
Hence, on the two-torus, our usual procedure of fixing the gauge by fixing the metric to that of flat space can even be carried out globally. It is then a simple matter to find a diffeomorphism of the form

$$\mathbb{1} = \boldsymbol{M}\,\boldsymbol{h}\,\boldsymbol{M}^{\mathrm{T}} \tag{75}$$

with $M^a{}_b = \frac{\partial\sigma^a}{\partial\xi^b}$, where the new coordinates ξ_1, ξ_2 are such that the metric again assumes unit form. The line element for a unit metric expressed i.t.o. complex coordinates $z = \sigma_1 + i\sigma_2$, $\bar{z} = \sigma_1 - i\sigma_2$ reads $\mathrm{d}s^2 = \mathrm{d}z\,\mathrm{d}\bar{z}$. Of course, there is no reason to assume that the new coordinates should still fulfill the original periodicity. Instead, $(\sigma_1,\sigma_2) \mathrel{\hat{=}} (\sigma_1,\sigma_2) + 2\pi(m,n)$, transforms under diffeomorphisms as

$$\xi_a \mathrel{\hat{=}} \xi_a + 2\pi(mu_a + nv_a), \qquad a \in \{1,2\}, \tag{76}$$

with worldsheet translations u_a, v_a. This means the equivalent values of σ_a form a lattice spanned by u_a and v_a. By rotating and rescaling the coordinate system, accompanied by a shift in ω to keep the metric normalized, we can achieve $\boldsymbol{u} = (1,0)$.



This leaves just two unfixed parameters between both the periodicity and the metric of the torus, namely the components of $\boldsymbol{v}$. Defining $\tau = v_1 + iv_2$, the periodicity becomes

$$z \mathrel{\hat{=}} z + 2\pi(m + n\tau). \tag{77}$$

To reproduce the original periodicity $(\sigma_1,\sigma_2) \mathrel{\hat{=}} (\sigma_1,\sigma_2) + 2\pi(m,n)$, we can absorb τ into the complex coordinate $z = \sigma_1 + \tau\sigma_2$ so that the line element becomes

$$\mathrm{d}s^2 = \mathrm{d}z\mathrm{d}\bar{z} = |\mathrm{d}\sigma_1 + \tau\mathrm{d}\sigma_2|^2. \tag{78}$$

Note: Tori that have different values for the Teichmüller parameter τ cannot in general be mapped into each other using a combination of diffeomorphisms and Weyl rescalings. Hence, there is a one-parameter family of conformally inequivalent tori. The integral over all metrics in the Polyakov path integral can be reduced to an integral over the single parameter τ by

[7] Incidentally, this is how the vacuum Einstein equations are derived from the action principle.

appropriate gauge fixing, but this final integral remains and has to be done by hand.

b), c) See part a).

d) Acting with S on the r.h.s. of eq. (67) yields

$$z + 2\pi(m + n\tau) \xrightarrow{S} z + 2\pi[m + n(\tau + 1)] = z + 2\pi[(m + n) + n\tau]. \tag{79}$$

Thus $S(m, n) = (m + n, n)$ establishes a one-to-one mapping from $\mathbb{Z}^2$ to $\mathbb{Z}^2$ and generates an equivalent set of identifications of points on the plane. What this means is that every τ differing from any other one by only an integer introduces redundancy, i.e. gauge freedom, into our description. We will find a way to remove it in part e) by restricting τ to a certain region of the τ-plane that contains only inequivalent values for the modulus.

Moving on, T does not leave the lattice invariant all on its own, but when combined with a rescaling $z \to \tau z$:

$$z + 2\pi(m + n\tau) \xrightarrow{T} z + 2\pi\Big(m - \frac{n}{\tau}\Big) \xrightarrow{z\to\tau z} \tau\Big[z + 2\pi\Big(m - \frac{n}{\tau}\Big)\Big] = \tau z + 2\pi(-n + \tau m). \tag{80}$$

Hence $T(m, n) = (-n, m)$ together with $z \to \tau z$ which leaves the theory invariant due its conformal symmetry also generates an equivalent lattice.

By the same arguments, combining U and $z \to (\tau + 1)z$

$$\begin{aligned} z + 2\pi(m + n\tau) \xrightarrow{U} z + 2\pi\Big(m + \frac{n\tau}{\tau + 1}\Big) &\xrightarrow{z\to(\tau+1)z} (\tau + 1)z + 2\pi\Big((\tau + 1)m + n\tau\Big) \\ &= (\tau + 1)z + 2\pi\Big(m + (m + n)\tau\Big) \end{aligned} \tag{81}$$

reveals that $U(m, n) = (m, m + n)$ and $z \to (\tau + 1)z$ combine to give a redundant lattice.

It is now easy to see that repeated applications of S, T (as mentioned above, we don't need U for this) reproduce an $SL(2, \mathbb{Z})$ transformation of the form (69). Without any particular order, we start with $p \in \mathbb{Z}$[8] applications of S,

$$\tau \xrightarrow{S^p} \tau + p, \tag{82}$$

followed by an application of T

$$\tau \xrightarrow{TS^p} \frac{-1}{\tau} + p = \frac{p\tau - 1}{\tau}. \tag{83}$$

Repeating the whole procedure, this time with $q \in \mathbb{Z}$ applications of S instead of p, we get

$$\tau \xrightarrow{TS^qTS^p} \frac{p\frac{q\tau-1}{\tau} - 1}{\frac{q\tau-1}{\tau}} = \frac{p(q\tau - 1) - \tau}{q\tau - 1} = \frac{(pq - 1)\tau - p}{q\tau - 1}. \tag{84}$$

To introduce a third parameter, we apply S another $r \in \mathbb{Z}$ times:

$$\tau \xrightarrow{S^rTS^qTS^p} \frac{(pq - 1)\tau + (pq - 1)r - p}{q\tau + qr - 1}. \tag{85}$$

Finally, the fourth parameter $s \in \mathbb{Z}$ comes from applying S^sT:

$$\tau \xrightarrow{TS^rTS^qTS^p} \frac{-(pq - 1)\frac{1}{\tau} + (pq - 1)r - p}{-q\frac{1}{\tau} + qr - 1} = \frac{-(pq - 1) + [(pq - 1)r - p]\tau}{-q + (qr - 1)\tau}, \tag{86}$$

$$\tau \xrightarrow{S^sTS^rTS^qTS^p} \frac{[(pq - 1)r - p]\tau + [(pq - 1)r - p]s - (pq - 1)}{(qr - 1)\tau + (qr - 1)s - q}. \tag{87}$$

[8]Naturally, S^p with $p < 0$ denotes p applications of the inverse of S, i.e. $S^{-1} : \tau \to \tau - 1$.

Thus by identifying

$$a = (pq-1)r - p, \qquad b = [(pq-1)r-p]s - (pq-1), \tag{88}$$
$$c = (qr-1), \qquad d = (qr-1)s - q, \tag{89}$$

and requiring

$$ab - cd = [(pq-1)r-p]\cdot\{[(pq-1)r-p]s-(pq-1)\} - (qr-1)\cdot[(qr-1)s-q] \overset{!}{=} 1, \tag{90}$$

which e.g. solved for p reads

$$p = \frac{1+q+s-rs-qrs}{q+s-qrs}, \tag{91}$$

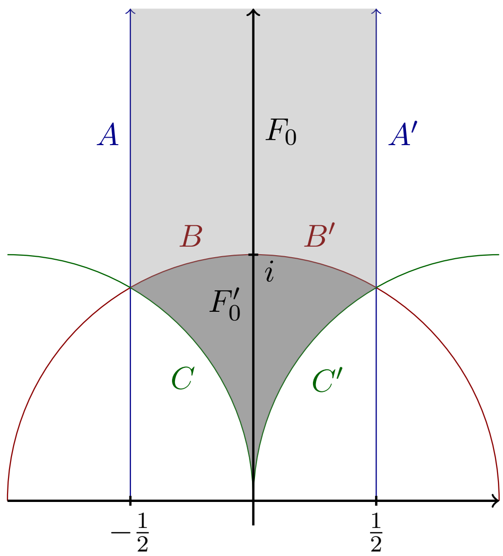
we see that repeated applications of S and T combine to give a general $SL(2,\mathbb{Z})$ transformation, i.e. any integer-valued 2×2-matrix with unit determinant, and hence generate the full group of special linear transformations.

However, the group on the τ-plane, i.e. the *modular group* of τ-transformations relating equivalent lattices is $SL(2,\mathbb{Z})/\mathbb{Z}_2 = PSL(2,\mathbb{Z})$, because

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \xrightarrow{\mathbb{Z}_2} \begin{pmatrix} -a & -b \\ -c & -d \end{pmatrix}. \tag{92}$$

and the transformation remains unchanged if all signs of a, b, c, d are reversed.

e) Starting with an arbitrary $\tau \in \mathbb{C}$, we can apply $T : \tau \to -\frac{1}{\tau}$ to map all τ with $|\tau| \leq 1$ to the area $|\tau| \geq 1$. We can further map all τ that lie outside $-\frac{1}{2} \leq \mathrm{Re}(\tau) \leq \frac{1}{2}$ to this strip by repeated application of $S : \tau \to \tau + 1$. Thus the fundamental domain F_0 is of the form of the lightly gray shaded area in the following illustration.



The edges A and A' are identified as are the parts B and B' of the green half circle.

> **Note:** An alternative fundamental region F_0' that is obtained from F_0 by an application of T is the dark gray area. Again, B and B' as well as C and C' are identified.