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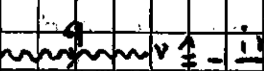
Quantum Field Theory II - Assignment 8

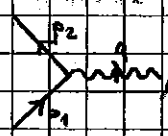
Problem 8.1 (Superficial degree of divergence for QED)

Review the Feynman rules of quantum electrodynamics and derive the superficial degree of divergence D for this theory. Proceed similarly to the derivation of D for the ϕ^4-theory as it was presented in the Lecture.

The superficial degree of divergence D is given by the power of momenta in the highest order, i.e. most divergent, contribution to an amplitude. Every loop yields a momentum integral $\int d^d k_i$ and therefore $d=4$ powers of momentum. All other powers derive from the propagators of internal lines. To specify these, we recall the QED Feynman rules for propagators

internal fermion line:  $\hat{=} \frac{i(\not p + m_0)}{p^2 - m_0^2} = \frac{i(\not p + m_0)}{(\not p + m_0)(\not p - m_0)} = \frac{i}{\not p - m_0}$

internal photon line:  $\hat{=} -\frac{i g_{\mu\nu}}{q^2}$

(vertex:  $\hat{=} -ie\gamma^\mu \delta^{(4)}(p_1 - p_2 - q)$)

We expect D to be a function of any of the following

$$D = D(E_f, I_f, E_\gamma, I_\gamma, V, L)$$

where the arguments are the numbers of

E_f: external fermions, I_f: internal fermions

E_γ: external photons, I_γ: internal photons

V: QED vertices, L: QED loops.

From the Feynman rules, we easily see the UV-structure of QED amplitudes is given by

$$A_{UV} \sim \int \frac{\prod_{l=1}^{L} d^4 k_l}{\prod_{i=1}^{I_f} (\not{p}_i - m_0) \cdot \prod_{j=1}^{I_\gamma} q_j^2},$$

resulting in a superficial degree of divergence of

$$D = 4L - I_f - 2I_\gamma$$

We proceed by inserting for the number of loop integrations

$$L = I_f + I_\gamma - (V-1),$$

which again follows from the Feynman rules as they attribute one momentum integral to each internal line $I = I_f + I_\gamma$ while each vertex contributes one $\delta^{(4)}$, one of whom only enforces overall momentum integration. Thus

$$D = 4(I_f + I_\gamma - (V-1)) - I_f - 2I_\gamma = 4 - 4V + 3I_f + 2I_\gamma$$

The number of vertices can also be reexpressed by

$$V = 2I_\gamma + E_\gamma = \frac{1}{2}(2I_f + E_f),$$

since every QED vertex has one photon- and two fermion-lines, and every internal line is connected to two vertices.
Altogether, we get

$$D = 4 - (2I_\gamma + E_\gamma) - 3\frac{1}{2}(2I_f + E_f) + 3I_f + 2I_\gamma$$

$$= 4 - E_\gamma - \frac{3}{2} E_f$$

Additional note: Looking at $\mathcal{L}_{QED} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \bar{\Psi}(i\gamma \cdot \partial - m_0)\Psi - e A_\mu \bar{\Psi} \gamma^\mu \Psi$, we see that since $[\Psi] = \frac{3}{2}$, $[A_\mu] = 1$, QED's coupling has mass-dimension, $[e] = 0$. From this, we conclude that QED is renormalizable in 4, i.e. has finitely many divergencies, but at every order in perturb.

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Problem 8.2 (One-loop structure of QED)

From problem 8.1, you should have found that the superficial degree of divergence of a QED-amplitude with an Interaction-Lagrangian $\mathcal{L}_{int} = \overline{\Psi}\gamma^\mu A_\mu \Psi$ is

$$D = 4 - E_\gamma - \tfrac{3}{2} E_f. \qquad (1)$$

Since $[e] = 0$, the theory is renormalizable.

a) Give D for each of the following amplitudes.

i) The vacuum energy.

$E_f = E_\gamma = 0$, $\quad D = 4 - E_\gamma - \frac{3}{2}E_f = 4$

ii) The photon propagator.

$E_f = 0,\ E_\gamma = 2$, $\quad D = 4 - E_\gamma = 2$

iii) The electron propagator.

$E_f = 2,\ E_\gamma = 0$, $\quad D = 4 - \frac{3}{2}E_f = 1$

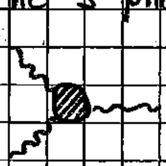
iv) The electron-electron-photon vertex.

$E_f = 2,\ E_\gamma = 1$, $\quad D = 4 - E_\gamma - \frac{3}{2}E_f = 0$

v) The 1-photon amplitude.

$E_f = 0,\ E_\gamma = 1$, $\quad D = 4 - E_\gamma = 3$

vi) The 3-photon amplitude.



$E_f = 0,\ E_\gamma = 3$, $\quad D = 4 - E_\gamma = 1$

vii) The 4-photon amplitude.

$E_f = 0,\ E_\gamma = 4$, $\quad D = 4 - E_\gamma = 0$

b) Verify explicitly that the one-loop diagram contributing to the one-point function vanishes using the discrete symmetry of charge conjugation, $j^\mu \to -j^\mu$, $A^\mu \to -A^\mu$. There are two Feynman diagrams contributing to the three-point function at one-loop order. Show that these cancel. Show that the diagrams contributing to any n-point photon amplitude, for n odd, cancel in pairs.

Charge conjugation $\hat{C}$ is a symmetry of QED, meaning

$$\hat{C}\,|\Omega\rangle = |\Omega\rangle$$

Since $j^\mu(x)$ changes sign under charge conjugation,

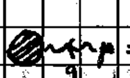
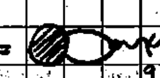
$$\hat{C}\, j^\mu(x)\, \hat{C}^\dagger = -j^\mu(x), \qquad \text{where} \quad \hat{C}^\dagger \hat{C} = \mathrm{id},$$

we infer that its vacuum expectation value vanishes at one-loop (and any other number of loops),

$$\langle \Omega | T\, j^\mu(x) | \Omega \rangle = \langle \Omega | j^\mu(x) | \Omega \rangle = \langle \Omega | \hat{C}^\dagger \hat{C}\, j^\mu(x)\, \hat{C}^\dagger \hat{C} | \Omega \rangle$$

$$= -\langle \Omega | j^\mu(x) | \Omega \rangle = 0.$$

This carries over to the photon one-point function

 $=$  $= -ie \int d^4x\, e^{-iqx} \langle \Omega | T\, j^\mu(x) | \Omega \rangle = 0,$

where $j^\mu(x) = \bar{\Psi}(x)\, \gamma^\mu\, \Psi(x)$.

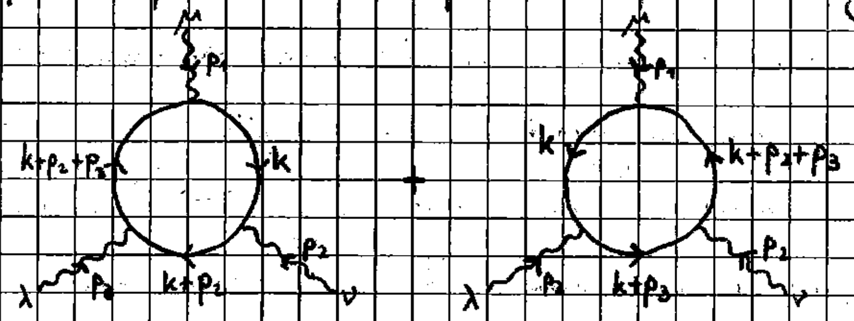
More explicitly, we may also write the one-point amplitude at one-loop as

$$i\Gamma^{(1)\mu}\Big|_{1\text{-loop}} = -ie \int \frac{d^4k}{(2\pi)^4} \frac{-i\,\mathrm{tr}[\gamma^\mu(\not{k}+m)]}{k^2-m^2} \overset{\mathrm{tr}(\gamma^\mu\gamma^\nu)=4\eta^{\mu\nu},\ \mathrm{tr}(\gamma^\mu)=0 \text{ see pr. 9.3, QFT I}}{=} -e \int \frac{d^4k}{(2\pi)^4} \frac{4k^\mu}{k^2-m^2} = 0,$$

which vanishes, since k^μ is odd and $\frac{1}{k^2-m^2}$ is even.

The three-point amplitude at one-loop consists of the two diagrams

$$i\Gamma^{(3)}\Big|^{\mu\nu\lambda}_{1\text{-loop}} =$$



$$= -(-ie)^3\int\frac{d^4k}{(2\pi)^4}\Big\{\mathrm{tr}\left[\gamma^\mu S_F(k)\gamma^\nu S_F(k+p_2)\gamma^\lambda S_F(k+p_2+p_3)\right]$$

$$+\,\mathrm{tr}\left[\gamma^\mu S_F(-k)\gamma^\nu S_F(-k-p_2)\gamma^\lambda S_F(-k-p_2-p_3)\right]\Big\},$$

where $S_F(k) := \frac{1}{\not{k}-m} = \frac{\not{k}+m}{k^2-m^2}$. To show that this vanishes, we use γ^μ's and S_F's behavior under charge conjugation,

$$\hat{C}^\dagger\gamma^\mu\hat{C} = (\gamma^\mu)^T, \quad \hat{C}^\dagger S_F(k)\hat{C} = -S_F(-k),$$

and the cyclicity of the trace. Thus

send to front using cyclic.

$$\mathrm{tr}\Big[\gamma^\mu\underbrace{\hat{C}\hat{C}^\dagger}_{1\!\!1} S_F(-k)\,\hat{C}\hat{C}^\dagger\gamma^\nu\hat{C}\hat{C}^\dagger S_F(-k-p_2)\,\hat{C}\hat{C}^\dagger\gamma^\lambda\hat{C}\hat{C}^\dagger S_F(-k-p_2-p_3)\,\hat{C}\hat{C}^\dagger\Big]$$

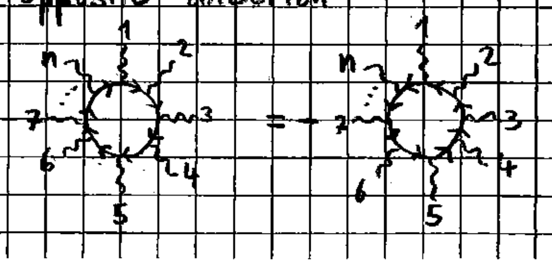
$$= \mathrm{tr}\Big[(\gamma^\mu)^T\left(-S_F(k)\right)(\gamma^\nu)^T\left(-S_F(k+p_2)\right)(\gamma^\lambda)^T\left(-S_F(k+p_2+p_3)\right)\Big]$$

$$= -\mathrm{tr}\Big[\gamma^\mu S_F(k)\gamma^\nu S_F(k+p_2)\gamma^\lambda S_F(k+p_2+p_3)\Big]$$

Reinserting into $i\Gamma^{(3)}|_{1\text{-loop}}$ yields immediately $i\Gamma^{(3)}|_{1\text{-loop}} = 0$.

We turn to the general case of all diagrams contributing to a photon n-point function where n is odd.

First note, that every fermionic loop with an odd number of vertices has a counterpart in which the momenta point along the loop in opposite direction

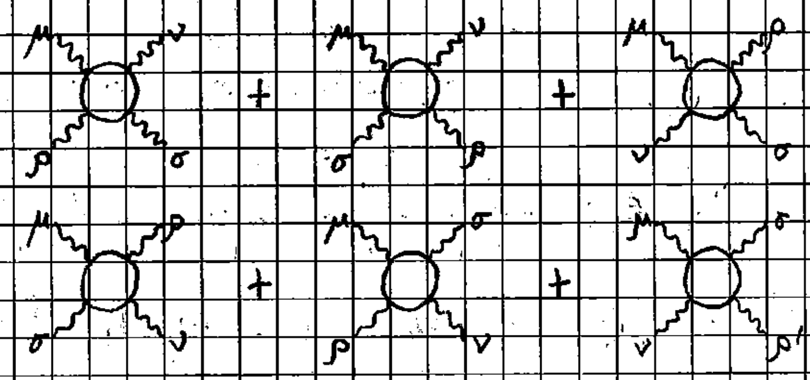


Since an n-point photon function receives contributions from $(n-1)!$, and $n!$ is even for $n>2$ (the case $n=1$, i.e. the one-loop diagram, we already showed to be vanishing in the beginning) we can always arrange all contributing diagrams in pairs of two, where for the special case of n odd, we may partner each diagram with its oppositely running counterpart in order to let each pair of our partition vanish individually.

c) The photon four-point amplitude is a sum of six diagrams. Show explicitly that the potential logarithmic divergences of these diagrams cancel.

The six diagrams are

$\Gamma^{(4)\mu\nu\rho\sigma} =$



where we omit momentum arrows in the fermion loops since clock- and counterclockwise running are equal for an even number of vertices.

The potential log-divergence of $i\Gamma^{(4)\mu\nu\rho\sigma}$ stems entirely from integrating over the internal momentum, i.e.

$$i\Gamma^{(4)\mu\nu\rho\sigma}_{div} \sim \int d^4k \, \frac{(\not k)^4}{(k^2)^4},$$

and is the same irrespective of a change in external momenta p_i, $i \in \{1,2,3,4\}$, or, for that matter, the fermion mass m. To speed up calculations, we may therefore set both to zero.

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The divergent part $i\Gamma^{(4)\mu\nu\rho\sigma}_{div}$ of the photon 4-point amplitude may then be written much more compactly as

$$i\Gamma^{(4)\mu\nu\rho\sigma}_{div} = \int \frac{d^4k}{(2\pi)^4}\frac{-1}{(k^2)^4}\Big[\mathrm{tr}(\gamma^\mu \not{k}\gamma^\nu \not{k}\gamma^\rho \not{k}\gamma^\sigma) + \mathrm{tr}(\gamma^\mu \not{k}\gamma^\nu \not{k}\gamma^\sigma \not{k}\gamma^\rho)$$
$$+\mathrm{tr}(\gamma^\mu \not{k}\gamma^\rho \not{k}\gamma^\nu \not{k}\gamma^\sigma) + \mathrm{tr}(\gamma^\mu \not{k}\gamma^\rho \not{k}\gamma^\sigma \not{k}\gamma^\nu)$$
$$+\mathrm{tr}(\gamma^\mu \not{k}\gamma^\sigma \not{k}\gamma^\nu \not{k}\gamma^\rho) + \mathrm{tr}(\gamma^\mu \not{k}\gamma^\sigma \not{k}\gamma^\rho \not{k}\gamma^\nu)\Big]$$

To get a grasp on this expression, we concern ourselves solely with the first trace initially. Using the gamma-matrices anti-commutation relations and some of the trace identities worked out in problem 9.3 of QFT I, it can be expressed as

$$\mathrm{tr}(\gamma^\mu \not{k}\gamma^\nu \not{k}\gamma^\rho \not{k}\gamma^\sigma) = 32\, k^\mu k^\nu k^\rho k^\sigma - 8k^2(k^\mu k^\nu g^{\rho\sigma} + k^\rho k^\sigma g^{\mu\nu}$$
$$+ k^\mu k^\sigma g^{\nu\rho} + k^\nu k^\rho g^{\mu\sigma}) + 4k^4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$$

We symmetrize products of k by sending $k^\mu k^\nu \to \frac{k^2}{4}g^{\mu\nu}$ and $k^\mu k^\nu k^\rho k^\sigma \longrightarrow \frac{k^4}{24}(g^{\mu\nu}g^{\rho\sigma} + g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$ to obtain

$$\mathrm{tr}(\gamma^\mu \not{k}\gamma^\nu \not{k}\gamma^\rho \not{k}\gamma^\sigma \not{k}) = \frac{4}{3}k^4(g^{\mu\nu}g^{\rho\sigma} - 2g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$$

From this expression, the other five traces follow simply by permutations of indices. Since the metric is symmetric, they all sum up to zero.