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Quantum Field Theory II - Assignment 6

Problem 6.1 (Path integral of the fermionic field with sources)

We define the generating functional for the correlation functions of the free Dirac theory as

$$Z_0[\bar{\eta},\eta] = \int \mathcal{D}\bar{\Psi}\,\mathcal{D}\Psi\, e^{i(S_0[\bar{\Psi},\Psi] + \bar{\eta}\cdot\Psi + \bar{\Psi}\cdot\eta)} \qquad (1)$$

with $S_0[\bar{\Psi},\Psi] = \int d^4x\, \bar{\Psi}(i\not{\partial} - m_0)\Psi$

a) Show by means of completing the square that (1) can be rearranged to

$$Z_0[\bar{\eta},\eta] = Z_0[\bar{\eta}=0,\eta=0]\, e^{-\bar{\eta}\cdot S_F\cdot\eta}, \qquad (2)$$

where S_F is the Feynman propagator of the free Dirac theory,

$$(i\not{\partial}_x - m_0 + i\epsilon)\, S_F(x-y) = i\delta^{(4)}(x-y).$$

Since we are working in a free theory, we may choose to shift not the time into the complex plane via $t \to t(1-i\epsilon)$ but instead make the replacement

$$\mathcal{H} \to \mathcal{H} - i\epsilon\,\phi^2(x), \qquad \text{i.e. } m_0^2 \to m_0^2 - i\epsilon, \qquad \text{(see eq. 7.80)}$$

in order to project the initial and final states to the vacuum. Thus, we may write the action as

$$S_0[\bar{\Psi},\Psi] \longrightarrow \int d^4x \int d^4y\, \bar{\Psi}\underbrace{(i\not{\partial} - m_0 + i\epsilon)\,\delta(x-y)}_{iS_F^{-1}(x-y)}\Psi = i\,\bar{\Psi}\cdot S_F^{-1}\cdot\Psi.$$

Completing the square yields

$$S_0[\bar{\Psi},\Psi] + \bar{\eta}\cdot\Psi + \bar{\Psi}\cdot\eta = i\,\bar{\Psi}\cdot S_F^{-1}\cdot\Psi + \bar{\eta}\cdot\Psi + \bar{\Psi}\cdot\eta$$

$$= (\bar{\Psi} - iS_F\bar{\eta})\cdot iS_F^{-1}\cdot(\Psi - iS_F\eta) + i\,\bar{\eta}\cdot S_F\cdot\eta$$

We define $\bar{\Psi}' = \bar{\Psi} - i S_F \bar{\eta}$ and $\Psi' = \Psi - i S_F \eta$ and transform our path integration to these new variables. Since we performed just a shift, the Jacobian of the transformation is in both cases just one. Thus, the integration measure remains invariant. Altogether, this gives

$$Z_0[\bar{\eta}, \eta] = \int \mathcal{D}\bar{\Psi}' \mathcal{D}\Psi' \, e^{i(\underbrace{\bar{\Psi}' \cdot i S_F^{-1} \cdot \Psi'}_{S_0[\bar{\Psi}', \Psi']} + i \bar{\eta} \cdot S_F \cdot \eta)}$$

$$= Z_0[\bar{\eta} = 0, \eta = 0] \, e^{-\bar{\eta} \cdot S_F \cdot \eta}$$

b) We add now an interaction part $\int d^4x \, \mathcal{L}_{int}(\bar{\Psi}, \Psi)$ to $S_0[\bar{\Psi}, \Psi]$. By proceeding as in the lecture for the bosonic case, show that the generating functional for the interacting theory can be written as

$$Z[\bar{\eta}, \eta] = Z_0[\bar{\eta} = 0, \eta = 0] \, e^{-\frac{\delta}{\delta \Psi} S_F \cdot \frac{\delta}{\delta \bar{\Psi}}} e^{i \int d^4x \, \mathcal{L}_{int}(\bar{\Psi}, \Psi)}$$

$$\cdot e^{i \bar{\eta} \cdot \Psi + i \bar{\Psi} \cdot \eta} \Big|_{\bar{\Psi} = \Psi = 0} . \qquad (3)$$

Our full action now reads

$$S[\bar{\Psi}, \Psi] = S_0[\bar{\Psi}, \Psi] + S_{int}[\bar{\Psi}, \Psi],$$

which makes our generating functional

$$Z[\bar{\eta}, \eta] := \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \, e^{i(S[\bar{\Psi}, \Psi] + \bar{\eta} \cdot \Psi + \bar{\Psi} \cdot \eta)}$$

$$= \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \, e^{i \int d^4x \, \mathcal{L}_{int}(\bar{\Psi}, \Psi)} e^{i(S_0[\bar{\Psi}, \Psi] + \bar{\eta} \cdot \Psi + \bar{\Psi} \cdot \eta)}$$

$$= e^{i \int d^4x \, \mathcal{L}_{int}(\frac{-\delta}{\delta \eta}, \frac{\delta}{i \delta \bar{\eta}})} \int \mathcal{D}\bar{\Psi} \mathcal{D}\Psi \, e^{i(S_0[\bar{\Psi}, \Psi] + \bar{\eta} \cdot \Psi + \bar{\Psi} \cdot \eta)},$$

where we used $F[\Psi] \, e^{i \bar{\eta} \cdot \Psi} = F[\frac{\delta}{i \delta \bar{\eta}}] \, e^{i \bar{\eta} \cdot \Psi}$ for any functional $F[\Psi]$.

Inserting our result from part a), we obtain

$$Z[\bar{\eta}, \bar{\eta}] = e^{i\int d^4x \mathcal{L}_{int}\left(\frac{\delta}{i\delta\eta}, \frac{\delta}{i\delta\bar{\eta}}\right)} Z_0[\bar{\eta}, \eta]$$

$$= Z_0[\bar{\eta} = 0, \eta = 0]\, e^{i\int d^4x \mathcal{L}_{int}\left(\frac{-\delta}{i\delta\eta}, \frac{\delta}{i\delta\bar{\eta}}\right)} e^{i\bar{\eta}\cdot S_F \cdot i\eta}$$

To proceed, we need to understand how to commute two Grassmann valued functionals containing derivatives. Note that

$$F\left[\frac{\delta}{i\delta\bar{\eta}}\right] G[\bar{\eta}]\, e^{i\bar{\eta}\cdot\psi} = F\left[\frac{\delta}{i\delta\bar{\eta}}\right] G\left[\frac{-\delta}{i\delta\psi}\right] e^{i\bar{\eta}\cdot\psi}$$

$$= (-1)^{\deg(F)\deg(G)} G\left[-\frac{\delta}{i\delta\psi}\right] F\left[\frac{\delta}{i\delta\bar{\eta}}\right] e^{i\bar{\eta}\cdot\psi} = (-1)^{\deg(F)\deg(G)} G\left[-\frac{\delta}{i\delta\psi}\right] F[\psi]\, e^{i\bar{\eta}\cdot\psi},$$

where we used eq. (7.239) to introduce a factor of $(-1)^{\deg(F)\deg(G)}$ with $\deg(F), \deg(G) \in \{0, 1\}$, depending on whether F and G are Grassmann even or odd, also called bosonic and fermionic, respectively. Since we are treating fermions here, ψ, $\bar{\psi}$, η, and $\bar{\eta}$ are all Grassmann odd. However, that makes both of the exponentials in the above expression for $Z[\bar{\eta}, \eta]$ Grassmann even. Thus

$$Z[\bar{\eta}, \eta] = Z_0[\bar{\eta} = 0, \eta = 0]\, e^{i\int d^4x \mathcal{L}_{int}\left(\frac{-\delta}{i\delta\eta}, \frac{\delta}{i\delta\bar{\eta}}\right)} e^{i\bar{\eta}\cdot S_F \cdot i\eta} \underbrace{e^{i(\bar{\eta}\cdot\psi + \bar{\psi}\cdot\eta)}\Big|_{\bar{\psi}=\psi=0}}_{1}$$

$$= Z_0[\bar{\eta} = 0, \eta = 0]\, e^{\frac{\delta}{\delta\psi}\cdot S_F \cdot \frac{\delta}{\delta\bar{\psi}}} e^{i\int d^4x \mathcal{L}_{int}(\bar{\psi}, \psi)} e^{i(\bar{\eta}\cdot\psi + \bar{\psi}\cdot\eta)}\Big|_{\bar{\psi}=\psi=0}$$

Problem 6.2 (Wick's theorem for fermions)

From eq. (3) one may derive that the correlation function for the free theory are given by

$$\langle T \prod_{i=1}^{n} \psi(x_i) \prod_{j=n+1}^{m+n} \bar{\psi}(x_j) \rangle = e^{\frac{\delta}{\delta\psi} S_F \cdot \frac{\delta}{\delta\bar{\psi}}} \prod_{i=1}^{n} \psi(x_i) \prod_{j=n+1}^{m+n} \bar{\psi}(x_j) \Big|_{\bar{\psi}=\psi=0} \quad (5)$$

a) Prove Wick's theorem for the free fermions, i.e. show that $\langle T \prod_{i=1}^{n} \psi(x_i) \prod_{j=n+1}^{m+n} \bar{\psi}(x_j) \rangle = 0$ for $n \neq m$ and

$$\langle T \prod_{i=1}^{n} \psi(x_i) \prod_{j=n+1}^{m+n} \bar{\psi}(x_j) \rangle = (-1)^{\frac{n}{2}(n-1)} S_F(x_1 - x_{n+1}) S_F(x_2 - x_{n+2}) \dots S_F(x_n - x_{2n})$$

+'all other contractions with appropriate signs', (6)

for $n = m$.

We can get a better grip on the correlation function in eq. (5) by writing the exponential as a series expansion,

$$\langle T \prod_{i=1}^{n} \psi(x_i) \prod_{j=n+1}^{m+n} \bar{\psi}(x_j) \rangle = \sum_{k=0}^{\infty} \frac{1}{k!} \left(-\frac{\delta}{\delta\psi} \cdot S_F \cdot \frac{\delta}{\delta\bar{\psi}}\right)^k \prod_{i=1}^{n} \psi(x_i) \prod_{j=n+1}^{m+n} \bar{\psi}(x_j) \Big|_{\bar{\psi}=\psi=0}$$

$$= \sum_{k=0}^{\min\{m,n\}} \frac{1}{k!} \left(-\frac{\delta}{\delta\psi} \cdot S_F \cdot \frac{\delta}{\delta\bar{\psi}}\right)^k \prod_{i=1}^{n} \psi(x_i) \prod_{j=n+1}^{m+n} \bar{\psi}(x_j) \Big|_{\bar{\psi}=\psi=0}$$

where the sum aborts at $\min\{m,n\}$ since $m+1$ functional derivatives acting on just m fields always gives zero.

If further $m \neq n$, then after applying all functional derivatives, either ψ-fields $(n>m)$ or $\bar{\psi}$-fields $(m>n)$ remain in every term appearing due to the product rule. Taking $\bar{\psi} = \psi = 0$ in the end makes the whole correlation function vanish. Physically, this reflects charge conservation.

On the other hand if $n = m$, we get

$$\langle T \prod_{i=1}^{n} \psi(x_i) \prod_{j=n+1}^{2n} \bar{\psi}(x_j) \rangle = \sum_{k=0}^{n} \frac{1}{k!} \left(\frac{\delta}{\delta\bar{\psi}} \cdot S_F \cdot \frac{\delta}{\delta\psi}\right)^k \prod_{i=1}^{n} \psi(x_i) \bar{\psi}(x_{n+i}) \Big|_{\bar{\psi}=\psi=0}$$

Again due to taking the limit $\bar\Psi = \Psi = 0$, in the end all terms of the sum over k disappear except for $k=n$, where we have an equal number of fields and derivatives. Thus

$$\langle T \prod_{i=1}^{n} \Psi(x_i) \prod_{j=n+1}^{2n} \bar\Psi(x_j) \rangle = \frac{1}{n!}\left(\frac{\delta}{\delta \bar\Psi} S_F \cdot \frac{\delta}{\delta \Psi}\right)^n \prod_{i=1}^{n} \Psi(x_i) \bar\Psi(x_{n+i})$$

$$= (-1)^{\frac{n}{2}(n-1)} S_F(x_1 - x_{n+1}) S_F(x_2 - x_{n+2}) \dots S_F(x_n - x_{2n})$$

$$+ \text{'all other contractions with appriate signs'.}$$

The last step actually requires lots of tedious combinatorics and commuting of Grassmann odd fields and derivatives.

b) Show by means of eq. (6) that

$$\langle T \bar\Psi_A(x) M^A{}_B \Psi^B(x) \bar\Psi_C(y) \hat{M}^C{}_D \Psi^D(y) \rangle = -\mathrm{Tr}\left(S_F(y-x) M S_F(x-y) \hat{M}\right)$$

$$+ \text{'uninteresting' } S_F(0)\text{-terms.}$$

To apply eq. (6), we note that in this case, $n=2$ and so $(-1)^{\frac{n}{2}(n-1)} = -1$.

$$\langle T \bar\Psi_A(x) M^A{}_B \Psi^B(x) \bar\Psi_C(y) \hat{M}^C{}_D \Psi^D(y) \rangle$$

$$= (-1)^3 M^A{}_B \hat{M}^C{}_D \langle T \Psi^B(x) \Psi^D(y) \bar\Psi_A(x) \bar\Psi_C(y) \rangle$$

$$= (-1)^3 M^A{}_B \hat{M}^C{}_D \left(-S_{F,A}^{B}(x-x) S_{F,C}^{D}(y-y) + S_{F,C}^{B}(x-y) S_{F,A}^{D}(y-x)\right)$$

$$= - S_{F,A}^{D}(y-x) M^A{}_B S_{F,C}^{B}(x-y) \hat{M}^C{}_D + S_{F,A}^{B}(0) M^A{}_B S_{F,C}^{D} \hat{M}^C{}_D$$

$$= -\mathrm{Tr}\left(S_F(y-x) M S_F(x-y) \hat{M}\right) + \text{'}S_F(0)\text{-terms'}$$