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Use a pen! (Tell me, why I shouldn't use a rubber? time) 12.05.2015

General Relativity - Exercise Sheet 5

1	2	3	4	Σ
15	9	15	5	44

Problem 1 (Curvature of a sphere) [15 points]

Last week, we calculated the Christoffel symbols

$$\Gamma^1_{22} = -\sin\theta\cos\theta, \quad \Gamma^2_{12} = \Gamma^2_{21} = \cot\theta,$$

on a two-sphere in $\mathbb{R}^3$ with fixed radius $r \in \mathbb{R}_+$ constant.
The metric was given by $\mathrm{diag}(\{g_{\mu\nu}\}) = (1, r^2, r^2\sin^2\theta)$.

a) Calculate all non-vanishing entries of the Riemann tensor $R^\mu{}_{\nu\alpha\beta}$.

Ignoring time, the spatial components of the Riemann curvature tensor in terms of the Christoffel symbols are given by

$$R^i{}_{jkm} := \partial_k \Gamma^i{}_{jm} - \partial_m \Gamma^i{}_{jk} + \Gamma^i{}_{nk}\Gamma^n{}_{jm} - \Gamma^i{}_{nm}\Gamma^n{}_{jk},$$

where $i,j,k,m,n \in \{x^1 = \theta, x^2 = \phi\}$. As mentioned in the lecture, the Riemann tensor has the symmetries

$$R_{ijkm} = -R_{jikm} = -R_{ijmk}, \quad R_{ijkm} = R_{kmij}, \quad R_{ijkm} + R_{ikmj} + R_{imjk} = 0$$

so that it only has $d = \frac{n^2}{12}(n^2-1) = \frac{4^2}{12}(4^2-1) = 20$ independent components.
Using all symmetries and the restriction to a 2-sphere, the Riemann tensor can be written more compactly as $R^i{}_{jkm} = \frac{1}{r^2}(\delta^i_k g_{jm} - \delta^i_m g_{jk})$. This form makes it plain that $R^i{}_{jkm} \neq 0$ only if $i=k$ or $i=m$, so that we may expect four non-zero entries. We compute them using the lecture's definition:

$$R^1{}_{212} = \partial_1 \Gamma^1{}_{22} - \partial_2 \underbrace{\Gamma^1{}_{21}}_{0} + \Gamma^1{}_{n2}\Gamma^n{}_{22} - \Gamma^1{}_{n2}\Gamma^n{}_{21}$$

$$= \partial_1 \Gamma^1{}_{22} + \Gamma^1{}_{22}\underbrace{\Gamma^2{}_{22}}_{0} - \Gamma^1{}_{22}\Gamma^2{}_{21}$$

$$= \partial_\theta(-\sin\theta\cos\theta) - (-\sin\theta\cos\theta)\cot\theta$$

$$= -\cos^2\theta + \sin^2\theta + \cos^2\theta = \sin^2\theta = -R^1{}_{221}$$ ✓

The other possibly non-zero independent entry of the Ricci-tensor can be found by looking at $i=2 \hat{=} \phi$, instead of 1.

$$R^2{}_{121} = \partial_2 \underbrace{\Gamma^2{}_{11}}_{0} - \partial_1 \Gamma^2{}_{12} + \Gamma^2{}_{n1} \underbrace{\Gamma^n{}_{11}}_{0\ \forall n} - \Gamma^2{}_{n1}\Gamma^n{}_{12}$$

$$= -\partial_1 \Gamma^2{}_{12} - \Gamma^2{}_{21}\Gamma^2{}_{12} = -\partial_\theta \cot\theta - \cot^2\theta$$

$$= -\frac{-\sin\theta}{\sin\theta} - \frac{-\cos^2\theta}{\sin^2\theta} - \cot^2\theta = 1 = -R^2{}_{112} \checkmark$$

b) Calculate the tensor Ricci-tensor $R_{\mu\nu}$.

The Ricci-tensor follows immediatly from the Riemann tensor by contracting the first and third indices, i.e.

$$R_{jm} = R^i{}_{jim}.$$

Since there is nothing to sum over here, contraction is trivial.

$$R_{11} = R^2{}_{121} = 1, \qquad R_{22} = R^1{}_{212} = \sin^2\theta \checkmark$$

All other entries, so that

$$\{R_{jm}\} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta \end{pmatrix} = \frac{1}{r^2}\{g_{jm}\}$$

c) Calculate the Ricci-scalar R.

The Ricci-scalar in turn follows directly from the Ricci-tensor by contraction with the metric g^{jm}.

$$R = g^{jm} R_{jm} = g^{jm}\frac{1}{r^2} g_{jm} = \frac{1}{r^2}\delta^m{}_m = \frac{2}{r^2} \checkmark$$

Problem 2 (Energy-momentum-tensor) [10 points]

Given the Lagrangian

$$L = \frac{1}{2} g^{\mu\nu} \partial_\mu \phi \, \partial_\nu \phi - V(\phi),$$

which describes a scalar field $\phi(x)$ obeying the Klein-Gordon equation, we're going to find the corresponding energy-momentum tensor.

a) Calculate $T^{\mu\nu}$ with

$$T^{\mu\nu} = -2 \frac{\delta L}{\delta g_{\mu\nu}} - L g^{\mu\nu}. \qquad \left(\text{Hint: } \delta g^{\mu\nu} = -g^{\mu\alpha} g^{\nu\beta} \delta g_{\alpha\beta}\right)$$

The functional derivative in $T^{\mu\nu}$ is given by

$$\frac{\delta L}{\delta g_{\mu\nu}} = \frac{\delta}{\delta g_{\mu\nu}} \left(\frac{1}{2} g^{\alpha\beta} \partial_\alpha \phi \, \partial_\beta \phi - V(\phi) \right) = \frac{1}{2} \partial_\alpha \phi \, \partial_\beta \phi \, \frac{\delta g^{\alpha\beta}}{\delta g_{\mu\nu}} \quad \checkmark$$

$$\overset{\text{hint}}{=} -\frac{1}{2} \partial_\alpha \phi \, \partial_\beta \phi \, g^{\rho\alpha} g^{\sigma\beta} \overbrace{\frac{\delta g_{\rho\sigma}}{\delta g_{\mu\nu}}}^{\delta_\rho^\mu \delta_\sigma^\nu} = -\frac{1}{2} \partial_\alpha \phi \, \partial_\beta \phi \, g^{\mu\alpha} g^{\nu\beta} = -\frac{1}{2} \partial^\mu \phi \, \partial^\nu \phi. \quad \checkmark$$

This yields the following energy-momentum tensor

$$T^{\mu\nu} = \partial^\mu \phi \, \partial^\nu \phi - g^{\mu\nu} L \quad \checkmark$$

b) Assume $\mathrm{diag}(\{g_{\mu\nu}\}) = (c^2, -a^2(t), -a^2(t), -a^2(t))$ and that ϕ is spatially constant $(\partial_i \phi = 0)$.

Identify the density ρ and the pressure p of the field ϕ using

$$\mathrm{diag}(\{T^\mu{}_\nu\}) = (\rho, -p, -p, -p).$$

What assumptions are made with this diagonal form of $T^\mu{}_\nu$?

We consider the quantity $T^\mu{}_\nu$ with one index up and one down to remove the distorting effects of the metric (modulo overall signs).

We arrive at $T^\mu{}_\nu$ by contracting one index with the metric, i.e. $T^\mu{}_\nu = g_{\nu\alpha} T^{\mu\alpha}$. From this, the energy density ρ and the pressure p of our scalar field ϕ can simply be read off,

$$\rho = T^0{}_0 = g_{0\alpha} T^{0\alpha} = g_{00} T^{00}, \quad \text{since the metric is diagonal}$$

$$= c^2\Big(\partial^0\phi\,\partial^0\phi - \underbrace{g^{00}}_{1/c^2} L\Big) = c^{-2}(\partial_t\phi)^2 - \frac{1}{2} g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi + V(\phi)$$

$$= c^{-2}(\partial_t\phi)^2 - \frac{1}{2}\partial_\mu\phi\,\partial^\mu\phi + V(\phi) = c^{-2}(\partial_t\phi)^2 - \frac{1}{2} g_{\mu\nu}\partial^\mu\phi\,\partial^\nu\phi + V(\phi)$$

$$= c^{-2}(\partial_t\phi)^2 - \frac{1}{2} g_{00}\,\partial^0\phi\,\partial^0\phi + V(\phi), \quad \text{since } \partial^i\phi = 0$$

$$= c^{-2}(\partial_t\phi)^2 - \frac{1}{2} c^{-2}(\partial_t\phi)^2 + V(\phi) = \frac{c^{-2}}{2}\dot\phi^2 + V(\phi) \quad \checkmark$$

$$p = -T^1{}_1 = -T^2{}_2 = -T^3{}_3 = -g_{1\alpha}T^{1\alpha} = -g_{11}T^{11}, \quad \text{again using the diagonal metric}$$

$$= -(-a^2(t))\Big(\underbrace{\partial^1\phi\,\partial^1\phi}_{0} - \underbrace{g^{11}}_{-1/a^2(t)} L\Big) = L = \frac{1}{2} g^{\mu\nu}\partial_\mu\phi\,\partial_\nu\phi - V(\phi)$$

$$= \frac{1}{2} g_{\mu\nu}\partial^\mu\phi\,\partial^\nu\phi - V(\phi) = \frac{1}{2} g_{00}\,\partial^0\phi\,\partial^0\phi - V(\phi), \quad \text{since } \partial^i\phi = 0$$

$$= \frac{c^{-2}}{2}(\partial_t\phi)^2 - V(\phi) = \frac{c^{-2}}{2}\dot\phi^2 - V(\phi) \quad \checkmark$$

The diagonal form of $T^\mu{}_\nu$ is known as Weyl's postulate which assumes a cosmos adequately described by a hydrodynamical model of an ideal fluid. In this model the cosmological principle, i.e. the notion that matter distribution in the Universe is isotropic and homogeneous on sufficiently large scales, justifies the diagonal form of $T^\mu{}_\nu$, which, as a result, lacks shear forces from the pressure component p^i in j-direction ($T^i{}_j$, $i \neq j$), energy flow per unit area in i-direction ($T^0{}_i$), and momentum densities in i-direction ($T^i{}_0$).

✓

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Problem 3 (An interesting line element II) [15 points]

On the last sheet, we calculated the Christoffel symbols $\Gamma^\alpha{}_{\mu\nu}$ of a line element ds with

$$ds^2 = c^2 dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta\, d\phi^2\right)\right].$$

a) Assume $k = 0$[2] and calculate the diagonal entries of the Ricci tensor $R_{\mu\mu}$. Remember,

$$R_{\mu\nu} = \partial_\alpha \Gamma^\alpha{}_{\mu\nu} - \partial_\nu \Gamma^\alpha{}_{\mu\alpha} + \Gamma^\beta{}_{\mu\nu}\Gamma^\alpha{}_{\beta\alpha} - \Gamma^\beta{}_{\mu\alpha}\Gamma^\alpha{}_{\beta\nu}.$$

For $k=0$, the above line element yields a total of 18 non-vanishing Christoffel symbols $\Gamma^\alpha{}_{\mu\nu}$. They are

$$\Gamma^0{}_{11} = a\dot a, \quad \Gamma^0{}_{22} = a\dot a r^2, \quad \Gamma^0{}_{33} = a\dot a r^2 \sin^2\theta$$

$$\Gamma^1{}_{10} = \Gamma^1{}_{01} = \frac{\dot a}{a}, \quad \Gamma^1{}_{22} = -r, \quad \Gamma^1{}_{33} = -r\sin^2\theta$$

$$\Gamma^2{}_{20} = \Gamma^2{}_{02} = \frac{\dot a}{a}, \quad \Gamma^2{}_{21} = \Gamma^2{}_{12} = \frac{1}{r}, \quad \Gamma^2{}_{33} = \sin\theta\cos\theta$$

$$\Gamma^3{}_{30} = \Gamma^3{}_{03} = \frac{\dot a}{a}, \quad \Gamma^3{}_{31} = \Gamma^3{}_{13} = \frac{1}{r}, \quad \Gamma^3{}_{32} = \Gamma^3{}_{23} = \cot\theta$$

They yield the following diagonal elements of the Ricci-tensor $R_{\mu\nu}$:

$$R_{00} = \partial_\alpha \underbrace{\Gamma^\alpha{}_{00}}_{0,\ \forall\alpha} - \partial_0 \Gamma^\alpha{}_{0\alpha} + \underbrace{\Gamma^\beta{}_{00}}_{0,\ \forall\beta}\Gamma^\alpha{}_{\beta\alpha} - \Gamma^\beta{}_{0\alpha}\Gamma^\alpha{}_{\beta 0}$$

$$= -\partial_0 \Gamma^i{}_{0i} - \Gamma^j{}_{0i}\Gamma^i{}_{j0} = -3\left(\partial_t \frac{\dot a}{a} + \frac{\dot a^2}{a^2}\right) = -3\left(\frac{\ddot a}{a} - \frac{\dot a^2}{a^2} + \frac{\dot a^2}{a^2}\right) = -3\frac{\ddot a}{a} \quad ✓$$

$$R_{11} = \partial_\alpha \Gamma^\alpha{}_{11} - \partial_1 \Gamma^\alpha{}_{1\alpha} + \Gamma^\beta{}_{11}\Gamma^\alpha{}_{\beta\alpha} - \Gamma^\beta{}_{1\alpha}\Gamma^\alpha{}_{\beta 1}$$

$$= \partial_0 \Gamma^0{}_{11} - \partial_1 \Gamma^1{}_{10} - \partial_1 \Gamma^2{}_{12} - \partial_1 \Gamma^3{}_{13} + \Gamma^0{}_{11}\underbrace{\Gamma^\alpha{}_{0\alpha}}_{3\Gamma^1{}_{01}} - \Gamma^0{}_{11}\Gamma^1{}_{01} - \Gamma^1{}_{10}\Gamma^0{}_{11}$$

$$- \Gamma^2{}_{12}\Gamma^2{}_{21} - \Gamma^3{}_{13}\Gamma^3{}_{31}$$

2: $k = 0 \hat{=}$ flat space

$$= \partial_t a\dot{a} - \partial_r \frac{\dot{a}}{a} - \partial_r \frac{1}{r} - \partial_r \frac{1}{r} + 3a\dot{a}\frac{\dot{a}}{a} - 2a\dot{a}\frac{\dot{a}}{a} - \frac{1}{r}\frac{1}{r} - \frac{1}{r}\frac{1}{r}$$

$$= \dot{a}^2 + a\ddot{a} + \frac{2}{r^2} + \dot{a}^2 - \frac{2}{r^2} = a\ddot{a} + 2\dot{a}^2 \quad \checkmark$$

$$R_{22} = \partial_\alpha \Gamma^\alpha{}_{22} - \partial_2 \Gamma^\alpha{}_{2\alpha} + \Gamma^\beta{}_{22}\Gamma^\alpha{}_{\beta\alpha} - \Gamma^\beta{}_{2\alpha}\Gamma^\alpha{}_{\beta 2}$$

$$= \partial_0 \Gamma^0{}_{22} + \partial_1 \Gamma^1{}_{22} - \partial_2 \Gamma^3{}_{23} + \Gamma^0{}_{22}\underbrace{\Gamma^\alpha{}_{0\alpha}}_{3\Gamma^1{}_{01}} + \Gamma^1{}_{22}\underbrace{\Gamma^\alpha{}_{1\alpha}}_{2\Gamma^2{}_{12}} - \Gamma^0{}_{22}\Gamma^2{}_{02}$$

$$- \Gamma^1{}_{22}\Gamma^2{}_{12} - \Gamma^2{}_{20}\Gamma^0{}_{22} - \Gamma^2{}_{21}\Gamma^1{}_{22} - \Gamma^3{}_{23}\Gamma^3{}_{32}$$

$$= \partial_t a\dot{a}r^2 + \partial_r(-r) - \partial_\theta \cot\theta + 3a\dot{a}r^2\frac{\dot{a}}{a} + 2(-r)\frac{1}{r} - a\dot{a}r^2\frac{\dot{a}}{a}$$

$$- (-r)\frac{1}{r} - \frac{\dot{a}}{a}a\dot{a}r^2 - \frac{1}{r}(-r) - \cot^2\theta$$

$$= \dot{a}^2r^2 + a\ddot{a}r^2 - 1 + 1 + \cot^2\theta + 3\dot{a}^2a^2 - 2 - \dot{a}^2r^2 + 1 - \dot{a}^2r^2 + 1 - \cot^2\theta$$

$$= a\ddot{a}r^2 + 2\dot{a}^2r^2 = (a\ddot{a} + 2\dot{a}^2)r^2 \quad \checkmark$$

$$R_{33} = \partial_\alpha \Gamma^\alpha{}_{33} - \partial_3 \underbrace{\Gamma^\alpha{}_{3\alpha}}_{0,\ \forall\alpha} + \Gamma^\beta{}_{33}\Gamma^\alpha{}_{\beta\alpha} - \Gamma^\beta{}_{3\alpha}\Gamma^\alpha{}_{\beta 3}$$

$$= \partial_0 \Gamma^0{}_{33} + \partial_1 \Gamma^1{}_{33} + \partial_2 \Gamma^2{}_{33} + \Gamma^0{}_{33}\underbrace{\Gamma^\alpha{}_{0\alpha}}_{3\Gamma^1{}_{01}} + \Gamma^1{}_{33}\underbrace{\Gamma^\alpha{}_{1\alpha}}_{2\Gamma^2{}_{12}} + \Gamma^2{}_{33}\underbrace{\Gamma^\alpha{}_{2\alpha}}_{\Gamma^3{}_{23}} - \Gamma^0{}_{33}\Gamma^3{}_{30}$$

$$- \Gamma^1{}_{33}\Gamma^3{}_{31} - \Gamma^2{}_{33}\Gamma^3{}_{32} - \Gamma^3{}_{30}\Gamma^0{}_{33} - \Gamma^3{}_{31}\Gamma^1{}_{33} - \Gamma^3{}_{32}\Gamma^2{}_{33}$$

$$= \partial_t a\dot{a}r^2\sin^2\theta + \partial_r(-r\sin^2\theta) + \partial_\theta(-\sin\theta\cos\theta) + 3a\dot{a}r^2\sin^2\theta\frac{\dot{a}}{a}$$

$$+ 2(-r\sin^2\theta)\frac{1}{r} + (-\sin\theta\cos\theta)\cot\theta - a\dot{a}r^2\sin^2\theta\frac{\dot{a}}{a} - (-r\sin^2\theta)\frac{1}{r}$$

$$- (-\sin\theta\cos\theta)\cot\theta - \frac{\dot{a}}{a}a\dot{a}r^2\sin^2\theta - \frac{1}{r}(-r\sin^2\theta) - \cot\theta(-\sin\theta\cos\theta)$$

$$= \dot{a}^2r^2\sin^2\theta + a\ddot{a}r^2\sin^2\theta - \sin^2\theta - \cos^2\theta + \sin^2\theta + 3\dot{a}^2r^2\sin^2\theta - 2\sin^2\theta$$

$$- \cos^2\theta - \dot{a}^2r^2\sin^2\theta + \sin^2\theta + \cos^2\theta - \dot{a}^2r^2\sin^2\theta + \sin^2\theta + \cos^2\theta$$

$$= 2\dot{a}^2r^2\sin^2\theta + a\ddot{a}r^2\sin^2\theta = (a\ddot{a} + 2\dot{a}^2)r^2\sin^2\theta \quad \checkmark$$

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b) Calculate the Ricci-scalar $R = g^{\mu\nu} R_{\mu\nu}$.

$$R = g^{\mu\nu} R_{\mu\nu} = g^{\mu\mu} R_{\mu\mu}, \quad \text{since } g^{\mu\nu} \text{ is diagonal}$$

$$= g^{00} R_{00} + g^{11} R_{11} + g^{22} R_{22} + g^{33} R_{33},$$

where $g^{00} R_{00} = R_{00} = -3\frac{\ddot{a}}{a}$

$$g^{11} R_{11} = -\frac{1}{a^2}\left(a\ddot{a} + 2\dot{a}^2\right) = -\frac{\ddot{a}}{a} - 2\frac{\dot{a}^2}{a^2}.$$

$$g^{22} R_{22} = -\frac{1}{a^2 r^2}\left(a\ddot{a} + 2\dot{a}^2\right) r^2 = -\frac{\ddot{a}}{a} - 2\frac{\dot{a}^2}{a^2}$$

$$g^{33} R_{33} = -\frac{1}{a^2 r^2 \sin^2\theta}\left(a\ddot{a} + 2\dot{a}^2\right) r^2 \sin^2\theta = -\frac{\ddot{a}}{a} - 2\frac{\dot{a}^2}{a^2}$$

$$\Longrightarrow R = -3\frac{\ddot{a}}{a} + 3\left(-\frac{\ddot{a}}{a} - 2\frac{\dot{a}^2}{a^2}\right) = -6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) \checkmark$$

c) Assuming once more that $\mathrm{diag}(\{T^\mu{}_\nu\}) = (\rho, -p, -p, -p)$, write down the 00 Einstein field equation,

$$G_{00} + g_{00}\Lambda = \frac{8\pi G}{c^4} T_{00}.$$

The Einstein tensor $G_{\mu\nu}$ is defined as

$$G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R,$$

so that the Einstein field equations can be expanded into

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + g_{\mu\nu}\Lambda = \frac{8\pi G}{c^4} T_{\mu\nu}.$$

Inserting our results from parts a) and b), the 00 equation reads

$$-3\frac{\ddot{a}}{a} - \frac{1}{2}\left[-6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right)\right] + c^2\Lambda = 3\frac{\dot{a}^2}{a^2} + c^2\Lambda = \frac{8\pi G}{c^4} c^2 \rho$$

$$\Longleftrightarrow \frac{\dot{a}^2}{a^2} = \frac{8\pi G}{3c^2}\rho - \frac{c^2}{3}\Lambda \checkmark$$

Problem 4 (Extraquestion: Spatial curvature of the universe) [5 points]

How would you go about designing an experiment that tests the flatness of the Universe as a whole? Can you distinguish between a flat and a non-flat cosmos?

Knowing the shape of the universe requires the consideration of two aspests:

1. its local geometry, which concerns the curvature of the observable Universe, and
2. its global geometry, i.e. the topology of the Universe as a whole.

To answer the second question: only if the observable Universe encompasses the entire Universe, might we be able to determine its global structure by observation.

As for a test resolving the question of the Universe's flatness, Charles Alcock realized decades ago that with astronomy's common method of obtaining distant object's separation from us by measuring the redshift of light reaching us from these objects, it would be possible to infer the Universe's geometry by observing any known-to-be spherical distribution of objects.

By analyzing any apparent distortion of that sphere and adapting the parameters that convert redshift to position so that the thus corrected distribution again becomes spherical, it would be possible to determine the geometry governing our Universe.