String Theory

Solution to Assignment 5

Janosh Riebesell

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Lecturer: Timo Weigand

1 String spectrum on D-branes

Consider the following setup of D-branes (all present at the same time) in bosonic string theory

- one spacetime-filling D25-brane,
- a stack of five coincident D12-branes along directions $\mu \in \{0, 1, \ldots, 11, 12\}$,
- a stack of seven coincident D5-branes along directions $\mu \in \{0, 1, 2, 12, 13, 14\}$,

where you should assume an arbitrary separation of the second and third brane stack along the common (DD) directions. Describe the open string spectrum up to the first excited level. The discussion should include an account of the mass of the states, the Chan-Paton factors and the gauge group along the various D-branes. What happens to the gauge group along the second brane as you separate the stack of coincident branes into 5 individual branes at non-zero relative distance?

The permeation of space by this arrangement of D-branes may be visualised with the following table which colours dimensions filled by branes (the darker the cell, the higher-dimensional the brane) and gives the kind and number of branes present in each dimension.

d	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
D25	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
D12	5	5	5	5	5	5	5	5	5	5	5	5	5													
D5	7	7	7										7	7	7											

We will describe the complete open string spectrum that emerges given the above D-brane constellation up to first excited level by giving all states $|\phi_i\rangle$, their masses M_i^2 and the normal ordering constants a_i for each possible combination of boundary conditions.
Before getting into close action, however, we need to prepare the necessary machinery.

- The states are the easiest: We simply start from the vacuum and act on it with modes α_i^μ corresponding to the excitation level, i.e. $i \in \{0, 1\}$ since we only investigate to first level, and contracted with gauge fields A_i^μ corresponding to the gauge group of the D-brane stack.
- To calculate the mass of a state, the relation

$$\alpha_n^- = \frac{1}{\sqrt{2\alpha' p^+}} \frac{1}{2} \sum_{i=1}^{D-2} \sum_{m \in \mathbb{Z}} \alpha_{n-m}^i \alpha_m^i, \tag{1}$$

comes in mighty handy. We proved this statement for the open string subject to pure Neumann boundary conditions in exercise 3 on assignment 4. For brevity, we will perform the following derivation with this limitation and then generalize to mixed boundary conditions at the end. For a metric with signature $(-1, 1, \ldots, 1)$, we know that M^2 equals $-\boldsymbol{p}^2$. In lightcone gauge, where the metric's components become $\eta_{+-} = -1 = \eta_{-+}$, $\eta_{ij} = \delta_{ij}$, $i, j \in \{1, \ldots, D-2\}$, $\boldsymbol{p}^2$ can be written as

$$M^2 = -\boldsymbol{p}^2 = 2p^+p^- - \sum_{i=1}^{D-2} p^i p^i. \tag{2}$$

Using $\alpha_0^\mu = \sqrt{2\alpha'}p^\mu$, eq. (2) can be rewritten as

$$\begin{aligned} M^2 &= \frac{2}{\sqrt{2\alpha'}} p^+ \alpha_0^- - \frac{1}{2\alpha'} \sum_{i=1}^{D-2} \alpha_0^i \alpha_0^i \\ &\overset{(1)}{=} \frac{1}{2\alpha'} \sum_{i=1}^{D-2} \sum_{m\in\mathbb{Z}} \alpha_{-m}^i \alpha_m^i - \frac{1}{2\alpha'} \sum_{i=1}^{D-2} \alpha_0^i \alpha_0^i \\ &= \frac{1}{2\alpha'} \sum_{i=1}^{D-2} \Big(\sum_{m\in\mathbb{Z}} \mathcal{N}(\alpha_{-m}^i \alpha_m^i) - 2a_\perp - \alpha_0^i \alpha_0^i \Big), \end{aligned} \tag{3}$$

where we reintroduced normal ordering and compensate by subtracting the normal ordering constant $a_\perp$ in lightcone gauge as it is precisely defined by the sum over the commutator of all transverse modes,

$$a_\perp = -\frac{1}{2} \sum_{i=1}^{D-2} \sum_{m<0} \underbrace{[\alpha_{-m}^i \alpha_m^i]}_{-m\,\eta^{ii}\delta_{-m,-m}} = -\frac{D-2}{2} \underbrace{\sum_{m>0} m}_{\zeta(-1)=-\frac{1}{12}} = \frac{D-2}{24}. \tag{4}$$

Now the normal-ordered sum can be written as

$$\sum_{m\in\mathbb{Z}} \mathcal{N}(\alpha_{-m}^i \alpha_m^i) = \sum_{m<0} \alpha_m^i \alpha_{-m}^i + \alpha_0^i \alpha_0^i + \sum_{m>0} \alpha_{-m}^i \alpha_m^i = \alpha_0^i \alpha_0^i + 2 \sum_{m>0} \alpha_{-m}^i \alpha_m^i. \tag{5}$$

Reinsertion into eq. (3) yields

$$M^2 = \frac{1}{\alpha'} \sum_{i=1}^{D-2} \sum_{m>0} \alpha_{-m}^i \alpha_m^i - \frac{a_\perp}{\alpha'} \equiv \frac{1}{\alpha'}(N_\perp - a_\perp), \tag{6}$$

where we defined the number operator in lightcone gauge $N_\perp$ which only counts excitations of physical degrees of freedom, i.e. excluding excitations in $X^\pm$.

- For a string with Neumann boundary conditions in all dimensions, $N_\perp$ takes the simple form as given above, i.e. $N_\perp = \sum_{i=1}^{D-2} \sum_{m>0} \alpha_{-m}^i \alpha_m^i$. More generally though, we may have any type of boundary condition in any dimension. If we let the indices i, j, k, and l count each type of boundary, i.e.

$$\left.\begin{matrix} \text{NN} \\ \text{DD} \\ \text{ND} \\ \text{DN} \end{matrix}\right\} \text{boundary conditions in dimensions} \left\{\begin{matrix} \{X^+, X^-, X^i | i \in I\} & \text{with } |I| = n_{\text{NN}}, ^1 \\ \{X^j | j \in J\} & \text{with } |J| = n_{\text{DD}}, \\ \{X^k | k \in K\} & \text{with } |K| = n_{\text{ND}}, \\ \{X^l | l \in L\} & \text{with } |L| = n_{\text{DN}}, \end{matrix}\right. \tag{7}$$

then $N_\perp$ generalizes straightforwardly to

$$N_\perp = \sum_{m\in\mathbb{N}} \Big[\sum_{i\in I} \alpha_{-m}^i \alpha_m^i + \sum_{j\in J} \alpha_{-m}^j \alpha_m^j \Big] + \sum_{n\in\mathbb{N}_0+\frac{1}{2}} \Big[\sum_{k\in K} \alpha_{-n}^k \alpha_n^k + \sum_{l\in L} \alpha_{-n}^l \alpha_n^l \Big]. \tag{8}$$

[1] $|I|$ denotes the cardinality of the set I, i.e. the number of elements in I.

- The normal ordering constant needs to be modified as well. We saw in eq. (4) that $a_\perp$ receives a contribution of $\frac{1}{24}$ for every dimension with NN boundary conditions. The same holds for DD boundary conditions. Due to the string field being half-integer moded in ND and DN dimensions, a careful calculation involving zeta function renormalization shows that each such dimension enters $a_\perp$ with a contribution of $-\frac{1}{48}$. Thus altogether, we get

$$\begin{aligned} a_\perp &= \frac{n_{\mathrm{NN}} + n_{\mathrm{DD}}}{24} - \frac{n_{\mathrm{ND}} + n_{\mathrm{DN}}}{48} = \frac{n_{\mathrm{NN}} + n_{\mathrm{DD}} + n_{\mathrm{ND}} + n_{\mathrm{DN}}}{24} - \frac{n_{\mathrm{ND}} + n_{\mathrm{DN}}}{16} \\ &= \frac{D-2}{24} - \frac{n_{\mathrm{ND}} + n_{\mathrm{DN}}}{16}. \end{aligned} \tag{9}$$

- There is one more contribution to the string's mass we need to consider if it is no longer free to move but attached to D-branes: its tension. For every dimension X^j, $j \in J$ with Dirichlet boundary conditions at both ends, the string's mass receives a contribution that grows linearly with the connected branes' separation $\Delta x_j = x_{l,j} - x_{0,j}$. The total string rest mass for an arbitrary constellation of boundary conditions therefore reads

$$M^2 = \frac{1}{\alpha'}(N_\perp - a_\perp) + T^2 \sum_{j \in J} \Delta x_j^2, \tag{10}$$

where $N_\perp$ and $a_\perp$ are now the ones from eqs. (8) and (9), respectively. This formula for M^2 incorporates everything: a shifted normal ordering constant, energy stored in strain, and a number operator that includes contributions from NN, DD, ND, and DN dimensions.

We are finally in a position to actually tackle the open string spectrum. Since each string has two ends, there are a total of $3^2 = 9$ different combinations possible with the above three types of D-branes present. We will treat each case in turn (and assume $D = 26$ in the sequel).

1. **D25-D25 strings** have NN boundary conditions in all dimensions, meaning $n_{\mathrm{ND}} = n_{\mathrm{DN}} = 0$ and eq. (9) gives the familiar result $a_\perp = 1$.
   - The **ground state** $|\mathrm{GS}\rangle$ is precisely the vacuum $|0, \boldsymbol{p}\rangle$, defined by the condition $\alpha_n^i |0, \boldsymbol{p}\rangle = 0 \ \forall n > 0$. Since this implies $N_\perp |0, \boldsymbol{p}\rangle = 0$, its mass is just

     $$M^2 |0, \boldsymbol{p}\rangle \overset{(10)}{=} -\frac{a_\perp}{\alpha'} |0, \boldsymbol{p}\rangle = -\frac{1}{\alpha'} |0, \boldsymbol{p}\rangle. \tag{11}$$

     The pure Neumann ground state is thus tachyonic. Its corresponding field transforms as a Lorentz scalar on the brane.[2] This counts for all the other ground states we will consider as well and will not be reiterated each time.
   - We get **first level excited states** by acting on the ground state with the lowest possible creation operator α_{-1}^i. Since the string lives on a spacetime-filling D-brane, the index i enumerates all transverse dimensions, $i \in I = \{1, 2, \ldots, D-2\}$ resulting in 24 linearly independent excitations $\alpha_{-1}^i |0, \boldsymbol{p}\rangle$. A general state at first excited level (L1) is a linear combination of all these basis states:

     $$|\mathrm{L1}\rangle = \sum_{i=1}^{D-2} A_i \, \alpha_{-1}^i |0, \boldsymbol{p}\rangle. \tag{12}$$

     Acting with $N_\perp$ on any of the basis states gives

     $$\begin{aligned} N_\perp \alpha_{-1}^i |0, \boldsymbol{p}\rangle &= \sum_{h \in I} \sum_{m \in \mathbb{N}} \alpha_{-m}^h \alpha_{-1}^h \alpha_m^i |0, \boldsymbol{p}\rangle = \alpha_{-1}^i \alpha_1^i \alpha_{-1}^i |0, \boldsymbol{p}\rangle \\ &= \alpha_{-1}^i (\alpha_{-1}^i \alpha_1^i + \underbrace{[\alpha_1^i, \alpha_{-1}^i]}_{\eta^{ii} \delta_{1,-(-1)}}) |0, \boldsymbol{p}\rangle = \alpha_{-1}^i |0, \boldsymbol{p}\rangle, \qquad \text{(no sum over } i\text{)}. \end{aligned} \tag{13}$$

[2] For brevity, transformational behaviors will not be shown here.

Equation (13) also holds for $|\mathrm{L1}\rangle$. This results in states at first excited level carrying a mass of

$$M^2|\mathrm{L1}\rangle = \frac{1}{\alpha'}(N_\perp - a_\perp)|\mathrm{L1}\rangle = \frac{1}{\alpha'}(1-1)|\mathrm{L1}\rangle = 0. \tag{14}$$

This entire level of the string spectrum is massless. Note further that the vector field A_μ transforms as a $U(1)$ gauge field on the D25 brane. This suggests a remarkable result: The open string spectrum includes photon states! Our entire approach to string theory started from the Polyakov action. It carries absolutely no hint of electromagnetic gauge invariance. Nevertheless, open strings exhibit Maxwellian field excitations. This result is entirely due to the normal ordering mass shift we encountered when passing from the classical to the quantum theory.

2. **D12-D12 strings** have NN boundary conditions along $X^0, X^1, \ldots, X^{12}$ and DD boundary conditions along $X^{13}, X^{14}, \ldots, X^{25}$. Thus, n_{ND} and n_{DN} remain zero and the normal ordering constant is still $a = 1$.

We face a new problem now: The exercise states that we are dealing with a stack of five coincident D12-branes. String states connected to coincident branes need an extra container of information to remember on which of the multiple overlapping branes a string starts and ends. A general basis of string states therefore needs to incorporate labels for

1. the Fock state $|n\rangle$ (where n is just the oscillation number, e.g. 0 in the case of the vacuum),
2. the momentum vector $\boldsymbol{p}$ (more precisely those components of $\boldsymbol{p}$ that lie in the NN dimensions; since all dimensions containing even one Dirichlet boundary are fixed at a particular point in space, they necessarily carry zero net momentum), and finally
3. a pair of integers r, s, the so-called Chan-Paton factors, that run over the N coincident branes.

Assembling parts 1 to 3, a completely determined basis state would be of the form $|n, \boldsymbol{p}, r\,s\rangle$. To express an arbitrary string state in this basis, we need to linearly combine the basis vectors in a way that is compatible with the group theoretic structure of D-brane stacks. A stack of N coincident branes hosts a $U(N)$ gauge group for which we can construct a representation by introducing N^2 Hermitian $N\times$-matrices.[3] Using these so-called Chan-Paton matrices $\boldsymbol{\lambda}^{rs}$, which encode the charge of a string state, an arbitrary state can be expressed as the following linear combination of basis states $|0, \boldsymbol{p}, r\,s\rangle$,

$$|n, \boldsymbol{p}, \boldsymbol{\lambda}\rangle = \sum_{r,s=1}^{N} \boldsymbol{\lambda}^{rs}\, |n, \boldsymbol{p}, r\,s\rangle. \tag{15}$$

- Employing this notation, the D12-D12 **ground state** can be written as

$$|GS\rangle = \sum_{r,s=1}^{5} \boldsymbol{\lambda}^{rs}\, |0, \boldsymbol{p}, r\,s\rangle. \tag{16}$$

  Of course, the Chan-Paton matrices commute freely with the modes and hence also with the number operator, so we still have $N_\perp|0, \boldsymbol{p}, r\,s\rangle = 0$. The ground state's mass thus remains unchanged at

$$M^2|\mathrm{GS}\rangle \overset{(11)}{=} -\frac{1}{\alpha'}|\mathrm{GS}\rangle. \tag{17}$$

- The **first excited state** is given by

$$|\mathrm{L1}\rangle = \sum_{r,s=1}^{5} \left(\sum_{i=2}^{12} \boldsymbol{A}_i^{rs} \alpha_{-1}^i + \sum_{j=13}^{25} \boldsymbol{\phi}_j^{rs} \alpha_{-1}^j \right) |0, \boldsymbol{p}, r\,s\rangle, \tag{18}$$

[3] The fact that coincident branes father extra gauge groups should come as no surprise to us. After all, a gauge symmetry is just a redundancy in our description of the *actual* physics going on, and exchanging the Chan-Paton factors certainly does not alter anything with regard to the physical position of the string or its spectrum. Hence we are dealing with pure gauge information here.

and carries zero mass, both for excitations parallel and transverse to the brane,

$$M_A^2|\mathrm{L1}\rangle \overset{(14)}{=} 0, \qquad M_\phi^2|\mathrm{L1}\rangle \overset{(14)}{=} 0. \tag{19}$$

The parallel excitations $\boldsymbol{A}_i^{rs}$ are $5^2 = 25$ massless vector bosons that transform as $U(5)$ gauge fields on the coincident D12-branes. The orthogonal excitations $\boldsymbol{\phi}_j^{rs}$ form 25 sets of 13 scalars each (since $j \in \{13, \dots, 25\}$) that transform under the adjoint of $U(5)$. These are the Goldstone bosons associated with spontaneous breaking of the 26-dimensional Poincaré invariance by the D12-branes.

3. The **D5-D5 strings** live on 7 coincident branes, have NN boundary conditions in dimensions $X\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1})$ and X^i for $i \in I = \{1, 2, 12, 13, 14\}$, and DD boundary conditions else. Since there are still no ND or DN dimensions, the normal ordering constant is $a_\perp = 1$.

   - In complete analogy to the D12-D12 string states, the **ground state** is given by

   $$|GS\rangle = \sum_{r,s=1}^{7} \boldsymbol{\lambda}^{rs}\,|0, \boldsymbol{p}, r\,s\rangle, \tag{20}$$

   and carries mass $M^2|\mathrm{GS}\rangle \overset{(11)}{=} -\frac{1}{\alpha'}|\mathrm{GS}\rangle$. (Bear in mind that the p^i may be nonzero only for $i \in I$.)

   - A general state at **first excited level** is of the form

   $$|\mathrm{L1}\rangle = \sum_{r,s=1}^{7} \left(\sum_{i\in I_{\not=}} \boldsymbol{A}_i^{rs} \alpha_{-1}^i + \sum_{j\in J} \boldsymbol{\phi}_j^{rs} \alpha_{-1}^j \right) |0, \boldsymbol{p}, r\,s\rangle, \tag{21}$$

   and carries zero mass, $M_A^2|\mathrm{L1}\rangle = M_\phi^2|\mathrm{L1}\rangle \overset{(14)}{=} 0$. Again, the vectors $\boldsymbol{A}_i^{rs}$ describe $U(7)$ gauge fields whereas the $\boldsymbol{\phi}_j^{rs}$ are scalars (49 sets of 20 each) transforming under the adjoint representation of $U(7)$.

4. **D12-D25 (D25-D12) strings** have NN boundary conditions along X^i, $i \in I = \{0, 1, 2, \dots, 12\}$, and DN (ND) boundary conditions else. Finally, we get a result for $a_\perp$ that is different from 1, namely

   $$a_\perp = \frac{D-2}{24} - \frac{n_{\mathrm{ND}} + n_{\mathrm{DN}}}{16} = 1 - \frac{13}{16} = \frac{3}{16}. \tag{22}$$

   This automatically means there can be no massless states in this sector.

   - Since only one end of the string attaches to a stack of coincident branes, states are labelled with only one Chan-Paton factor. The **ground state** thus reads

   $$|GS\rangle = \sum_{r(s)=1}^{5} \boldsymbol{\lambda}^{r(s)}\,|0, \boldsymbol{p}, r\,(s)\rangle, \tag{23}$$

   with a mass of $M^2|GS\rangle = -\frac{a_\perp}{\alpha'}|GS\rangle = -\frac{3}{16\alpha'}|GS\rangle$.

   - Removing one Chan-Paton factor from eq. (21) and making the mode operator in the DN (ND) dimensions half-integer, the **first excited level** states read

   $$|\mathrm{L1}\rangle = \sum_{r(s)=1}^{5} \left(\sum_{i=2}^{12} \boldsymbol{A}_i^{r(s)} \alpha_{-1}^i + \sum_{l(k)=13}^{25} \boldsymbol{\phi}_{l(k)}^{r(s)} \alpha_{-\frac{1}{2}}^{l(k)} \right) |0, \boldsymbol{p}, r\,(s)\rangle, \tag{24}$$

   and carry a mass of

   $$\begin{aligned} M_A^2|\mathrm{L1}\rangle &= \frac{1}{\alpha'}(N_\perp^A - a_\perp)|\mathrm{L1}\rangle = \frac{1}{\alpha'}\left(1 - \frac{3}{16}\right)|\mathrm{L1}\rangle = \frac{13}{16}|\mathrm{L1}\rangle, \\ M_\psi^2|\mathrm{L1}\rangle &= \frac{1}{\alpha'}(N_\perp^\psi - a_\perp)|\mathrm{L1}\rangle = \frac{1}{\alpha'}\left(\frac{1}{2} - \frac{3}{16}\right)|\mathrm{L1}\rangle = \frac{5}{16}|\mathrm{L1}\rangle. \end{aligned} \tag{25}$$

The parallel excitations $A_i^{r(s)}$ are massive vector bosons transforming under the (anti-)fundamental representation of $U(5)$.[4] The orthogonal excitations $\psi_{l(k)}^{r(s)}$ are massive scalars in the fundamental representation of $U(7)$.

5. The **D5-D25 (D25-D5) strings** have NN boundary conditions in dimensions X^i, $i \in I = \{0, 1, 2, 12, 13, 14\}$, and DN (ND) boundary conditions in $X^{l(k)}$, $l(k) \in L(K) = \{3, 4, \ldots, 11, 15, 16, \ldots, 25\}$. That makes for a negative normal ordering constant of

$$a_\perp = 1 - \frac{20}{16} = -\frac{1}{4}. \tag{26}$$

 - As a direct consequence, the **ground state**

$$|GS\rangle = \sum_{r(s)=1}^{7} \lambda^{r(s)} \, |0, \boldsymbol{p}, r\,(s)\rangle, \tag{27}$$

 is no longer tachyonic,[5]

$$M^2 |GS\rangle = -\frac{a_\perp}{\alpha'} |GS\rangle = \frac{1}{4\alpha'} |GS\rangle. \tag{28}$$

 - And just like before, the **first excited level states**

$$|\mathrm{L1}\rangle = \sum_{r(s)=1}^{7} \left(\sum_{i \in I_{\not{t}}} A_i^{r(s)} \alpha_{-1}^i + \sum_{l(k) \in L(K)} \psi_{l(k)}^{r(s)} \alpha_{-\frac{1}{2}}^{l(k)} \right) |0, \boldsymbol{p}, r\,(s)\rangle, \tag{29}$$

 are all massive,

$$M_A^2 |\mathrm{L1}\rangle = \frac{1}{\alpha'} (N_\perp^A - a_\perp) |\mathrm{L1}\rangle = \frac{1}{\alpha'} \left(1 + \frac{1}{4} \right) |\mathrm{L1}\rangle = \frac{5\alpha'}{4} |\mathrm{L1}\rangle, \tag{30}$$

$$M_\psi^2 |\mathrm{L1}\rangle = \frac{1}{\alpha'} (N_\perp^\psi - a_\perp) |\mathrm{L1}\rangle = \frac{1}{\alpha'} \left(\frac{1}{2} + \frac{1}{4} \right) |\mathrm{L1}\rangle = \frac{3\alpha'}{4} |\mathrm{L1}\rangle. \tag{31}$$

 Once again, the $A_i^{r(s)}$ are in the (anti-)fundamental and the $\psi_{l(k)}^{r(s)}$ in the fundamental representation of $U(7)$.

6. The **D5-D12 (D12-D5) strings** have
   i. NN boundary conditions in dimensions X^0, X^1, X^2, and X^{12},
   ii. DD boundary conditions in X^j with $j \in J = \{15, 16, \ldots, 25\}$,
   iii. DN (ND) boundary conditions in $X^{l(k)}$ with $l(k) \in L(K) = \{3, 4, \ldots, 11\}$, and
   iv. ND (DN) boundary conditions in X^{13}, X^{14}.

 This interesting combination of branes yields a normal ordering constant of

$$a_\perp = \frac{D-2}{24} - \frac{n_{\mathrm{ND}} + n_{\mathrm{DN}}}{16} = 1 - \frac{11}{16} = \frac{5}{16}. \tag{32}$$

 - The ground state now carries two Chan-Paton factors. We let r run from 1 to 7 and s from 1 to 5. Then[6]

$$|GS\rangle = \sum_{r=1}^{7} \sum_{s=1}^{5} \boldsymbol{\lambda}^{rs} \, |0, \boldsymbol{p}, rs\,(sr)\rangle. \tag{33}$$

[4] The only difference between the fundamental and anti-fundamental vectors is that they are complex conjugates fields, i.e. they have opposite charge.
[5] It is, of course, still a scalar, though.
[6] Note that $\boldsymbol{p}$ may be non-zero only for p^0, p^1, p^2, p^{12}. The string field is thus effectively restrained to the intersection $\mathrm{D5} \cap \mathrm{D12}$.

In some dimensions, the string starts and ends on different D-branes with non-zero separation. The mass therefore receives a contribution from tension for the first time,

$$M^2|GS\rangle = \left(\frac{1}{\alpha'}(N_\perp - a_\perp) + T^2 \sum_{j\in J} \Delta x_j^2\right)|GS\rangle = \left(-\frac{5}{16\alpha'} + T^2 \sum_{j=15}^{25} \Delta x_j^2\right)|GS\rangle \tag{34}$$

- Writing down a state that captures all possible excitations at **first excited level** results in the somewhat unwieldy expression

$$|\mathrm{L1}\rangle = \sum_{r=1}^{7}\sum_{s=1}^{5}\Bigg(\underbrace{\sum_{i\in\{2,12\}} \boldsymbol{A}_i^{rs(sr)}\alpha_{-1}^i}_{\text{NN}} + \underbrace{\sum_{j=15}^{25} \boldsymbol{\phi}_j^{rs(sr)}\alpha_{-1}^j}_{\text{DD}} + \underbrace{\sum_{l(k)\in L(K)} \boldsymbol{\psi}_{l(k)}^{rs(sr)}\alpha_{-1}^{l(k)}}_{\text{DN (ND)}} + \underbrace{\sum_{k(l)\in\{13,14\}} \boldsymbol{\xi}_{k(l)}^{rs(sr)}\alpha_{-\frac{1}{2}}^{k(l)}}_{\text{ND (DN)}}\Bigg)|0,\boldsymbol{p},r\,(s)\rangle.$$

The mass spectrum is

$$M_A^2|\mathrm{L1}\rangle = M_\phi^2|\mathrm{L1}\rangle = \left(\frac{11}{16\alpha'} + T^2 \sum_{j=15}^{25} \Delta x_j^2\right)|\mathrm{L1}\rangle, \tag{35}$$

$$M_\psi^2|\mathrm{L1}\rangle = M_\xi^2|\mathrm{L1}\rangle = \left(\frac{3}{16\alpha'} + T^2 \sum_{j=15}^{25} \Delta x_j^2\right)|\mathrm{L1}\rangle. \tag{36}$$

Quoting the lecture notes (p. 168), the Chan-Paton factors reveal that all excitations transform under the (anti-)bifundamental[7] representation of the gauge group $U(5)\times U(7)$.

2 Orientifolds

In the lecture we have so far discussed only the so-called oriented string theory. This means that there is a clear distinction between left- and right-movers for the closed string and between the two endpoints at $\sigma = 0$ and $\sigma = l$ for the open theory.

a) Show that even after gauge fixing the classical worldsheet action is invariant under the parity transformation

$$\tau \to \tau, \qquad \sigma \to l - \sigma. \tag{37}$$

At the quantum level this so-called **orientifold symmetry** is implemented via a unitary operator Ω acting on the string field as

$$X^\mu(\tau,\sigma) \to \Omega^\dagger\, X^\mu(\tau,\sigma)\,\Omega = X^\mu(\tau, l-\sigma). \tag{38}$$

b) Show that this induces the following action in the modes of the string field

i)	closed	$\Omega^\dagger\,\alpha_{l,n}^\mu\,\Omega = \alpha_{r,n}^\mu,$	$\Omega^\dagger\,\alpha_{r,n}^\mu\,\Omega = \alpha_{l,n}^\mu,$	(39)
ii)	open NN	$\Omega^\dagger\,\alpha_n^\mu\,\Omega = (-1)^n\alpha_n^\mu,$		(40)
iii)	open DD	$\Omega^\dagger\,\alpha_n^\mu\,\Omega = (-1)^{n+1}\alpha_n^\mu,$	$\Omega^\dagger\,x_{0/l}\,\Omega = x_{0/l},$	(41)
iv)	open DN	$\Omega^\dagger\,\alpha_{n+\frac{1}{2}}^\mu\,\Omega = i(-1)^{n+1}\alpha_{n+\frac{1}{2}}^\mu.$		(42)

c) One can now define the unoriented or orientifolded string theory by keeping only those states in the string spectrum which are invariant under the above orientifold action. In other words, one considers the quotient theory of the original string theory by the $\mathbb{Z}_2$ action defined in eq. (39). To determine this quotient, one needs to know in addition the phase of the action of Ω on the

[7] This is an abbreviation for the fundamental × antifundamental (antifundamental × fundamental) representation.

vacuum. This is not fixed uniquely, but constrained by subtle quantum constraints (tadpole cancellation conditions) in the full theory which are beyond our techniques at this stage. In the closed string sector, the consistent phase of Ω acting on the vacuum is $+1$. For the open string it turns out that there are 2 possibilities: $+1$ and -1.

According to this logic, which states does the closed orientifolded string theory contain at the first excited level? For the respective phases $+1$ or -1, which excitation levels are kept for open strings ending on a single D-brane in the NN and DD sector?

d) Now consider the open string sector for a stack of N coincident D-branes. Depending on the phase, the orientifold action acts on the vacuum as

$$\Omega|0, \boldsymbol{p}, rs\rangle = \pm|0, \boldsymbol{p}, sr\rangle, \tag{43}$$

i.e. the Chan-Paton factors are exchanged. What does this mean for the states kept at the first excited level for the two signs? How many states are kept?

For group theory experts: Guess the corresponding gauge groups on the D-branes.

a) After gauge fixing to a flat worldsheet metric, the Polyakov action reduces to a set of D free scalar fields X^μ, $\mu \in \{0, \dots, D-1\}$,

$$S_\mathrm{P} = \frac{T}{2} \int_\Sigma \mathrm{d}\tau \mathrm{d}\sigma \Big((\partial_\tau X)^2 - (\partial_\sigma X)^2 \Big). \tag{44}$$

This integral remains invariant under worldsheet parity,

$$\begin{aligned} S_\mathrm{P} \xrightarrow[\sigma \to l-\sigma]{\tau \to \tau} S'_\mathrm{P} &= \frac{T}{2} \int_{-\infty}^{\infty} \mathrm{d}\tau \int_l^0 (-\mathrm{d}\sigma) \Big((\partial_\tau X)^2 - (-\partial_\sigma X)^2 \Big) \\ &= \frac{T}{2} \int_{-\infty}^{\infty} \mathrm{d}\tau \int_0^l \mathrm{d}\sigma \Big((\partial_\tau X)^2 - (\partial_\sigma X)^2 \Big) = S_\mathrm{P}. \end{aligned} \tag{45}$$

b) Since the Polyakov is unaffected by worldsheet parity, the same must be true for the string field's equation of motion $(\partial_\tau^2 - \partial_\sigma^2) X^\mu = 0$ (which it obviously is). But if the e.o.m. is unaltered, then its solution need not be modified either. Therefore, we may impose that the string field's mode expansion remains invariant under worldsheet parity.

i) The closed string mode expansion is very similar to that of the open string as derived in exercise 1 on assignment 3, except that with periodic boundary conditions, left- and right-moving oscillations are not reflected at $\sigma = 0$ or l. Instead they continuously propagate along the loop of the string without obstruction. To include both left- and right-moving excitations, the mode expansion therefore needs to contain two types of modes $\alpha_{l,n}^\mu$ and $\alpha_{r,n}^\mu$,

$$X_\mathrm{cl}^\mu(\tau, \sigma) = x_0^\mu + \frac{2\pi\alpha'}{l} p^\mu \tau + i\sqrt{\tfrac{\alpha'}{2}} \sum_{n \neq 0} \frac{\alpha_{l,n}^\mu}{n} e^{-\frac{2\pi i}{l} n(\tau+\sigma)} + \frac{\alpha_{r,n}^\mu}{n} e^{-\frac{2\pi i}{l} n(\tau-\sigma)}. \tag{46}$$

Acting with Ω on $X_\mathrm{cl}^\mu(\tau, \sigma)$, we get

$$\Omega^\dagger X_\mathrm{cl}^\mu(\tau, \sigma)\, \Omega = x_0^\mu + \frac{2\pi\alpha'}{l} p^\mu \tau + i\sqrt{\tfrac{\alpha'}{2}} \sum_{n \neq 0} \frac{\Omega^\dagger \alpha_{l,n}^\mu \Omega}{n} e^{-\frac{2\pi i}{l} n(\tau+\sigma)} + \frac{\Omega^\dagger \alpha_{r,n}^\mu \Omega}{n} e^{-\frac{2\pi i}{l} n(\tau-\sigma)}, \tag{47}$$

$$\parallel$$

$$X_\mathrm{cl}^\mu(\tau, l-\sigma) = x_0^\mu + \frac{2\pi\alpha'}{l} p^\mu \tau + i\sqrt{\tfrac{\alpha'}{2}} \sum_{n \neq 0} \frac{\alpha_{l,n}^\mu}{n} e^{-\frac{2\pi i}{l} n(\tau-\sigma)} \underbrace{e^{-2\pi i n}}_{1} + \frac{\alpha_{r,n}^\mu}{n} e^{-\frac{2\pi i}{l} n(\tau+\sigma)} \underbrace{e^{2\pi i n}}_{1}. \tag{48}$$

Equations (47) and (48) are equal only if

$$\Omega^\dagger\,\alpha^\mu_{l,n}\,\Omega = \alpha^\mu_{r,n}, \qquad \Omega^\dagger\,\alpha^\mu_{r,n}\,\Omega = \alpha^\mu_{l,n}. \tag{49}$$

ii) For the open NN string, Ω acts on the mode expansion as

$$\Omega^\dagger\,X^\mu_{\rm NN}(\tau,\sigma)\,\Omega = x^\mu_0 + \frac{2\pi\alpha'}{l}p^\mu\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\Omega^\dagger\,\alpha^\mu_n\,\Omega}{n}e^{-\frac{\pi i}{l}n\tau}\cos\left(\tfrac{\pi}{l}n\sigma\right), \tag{50}$$

$$\parallel$$

$$X^\mu_{\rm NN}(\tau,l-\sigma) = x^\mu_0 + \frac{2\pi\alpha'}{l}p^\mu\tau + i\sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}e^{-\frac{\pi i}{l}n\tau}\underbrace{\cos\left(\pi n - \tfrac{\pi}{l}n\sigma\right)}_{(-1)^n\cos\left(\frac{\pi}{l}n\sigma\right)}, \tag{51}$$

where we used $\cos(a-b) = \sin(a)\sin(b) + \cos(a)\cos(b)$. Invariance under worldsheet parity thus requires

$$\Omega^\dagger\,\alpha^\mu_n\,\Omega = (-1)^n\alpha^\mu_n. \tag{52}$$

iii) For the open DD string, Ω acts on the mode expansion as

$$\Omega^\dagger\,X^\mu_{\rm DD}(\tau,\sigma)\,\Omega = x^\mu_0 + (x^\mu_l - x^\mu_0)\frac{\sigma}{l} + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{\Omega^\dagger\,\alpha^\mu_n\,\Omega}{n}e^{-\frac{\pi i}{l}n\tau}\sin\left(\tfrac{\pi}{l}n\sigma\right), \tag{53}$$

$$\parallel$$

$$X^\mu_{\rm DD}(\tau,l-\sigma) = x^\mu_0 + \underbrace{(x^\mu_l - x^\mu_0)\frac{l-\sigma}{l}}_{x^\mu_l - x^\mu_0 + (x^\mu_0 - x^\mu_l)\frac{\sigma}{l}} + \sqrt{2\alpha'}\sum_{n\neq 0}\frac{\alpha^\mu_n}{n}e^{-\frac{\pi i}{l}n\tau}\underbrace{\sin\left(\pi n - \tfrac{\pi}{l}n\sigma\right)}_{(-1)^{n+1}\sin\left(\frac{\pi}{l}n\sigma\right)}, \tag{54}$$

where we used $\sin(a-b) = \sin(a)\cos(b) - \cos(a)\sin(b)$. Comparing eqs. (53) and (54), we conclude

$$\Omega^\dagger\,\alpha^\mu_n\,\Omega = (-1)^{n+1}\alpha^\mu_n \qquad \text{and} \qquad \Omega^\dagger\,x_{0/l}\,\Omega = x_{0/l}. \tag{55}$$

iv) Since Ω effectively turns the string around, it transforms DN into ND boundary conditions. The effect of Ω on the DN mode expansion is therefore

$$\Omega^\dagger\,X^\mu_{\rm DN}(\tau,\sigma)\,\Omega = x^\mu_0 - \sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\frac{\Omega^\dagger\,\alpha^\mu_{n+\frac{1}{2}}\,\Omega}{n+\frac{1}{2}}e^{-\frac{\pi i}{l}(n+\frac{1}{2})\tau}\sin\Big[\tfrac{\pi}{l}(n+\tfrac{1}{2})\sigma\Big], \tag{56}$$

$$\parallel$$

$$X^\mu_{\rm ND}(\tau,l-\sigma) = x^\mu_0 + i\sqrt{2\alpha'}\sum_{n\in\mathbb{Z}}\frac{\alpha^\mu_{n+\frac{1}{2}}}{n+\frac{1}{2}}e^{-\frac{\pi i}{l}(n+\frac{1}{2})\tau}\underbrace{\cos\Big[\pi(n+\tfrac{1}{2}) - \tfrac{\pi}{l}(n+\tfrac{1}{2})\sigma\Big]}_{(-1)^n\sin\left[\frac{\pi}{l}(n+\frac{1}{2})\sigma\right]}, \tag{57}$$

where we again used $\cos(a-b) = \sin(a)\sin(b) + \cos(a)\cos(b)$. For the right hand sides eqs. (56) and (57) to be equal, we need

$$\Omega^\dagger\,\alpha^\mu_{n+\frac{1}{2}}\,\Omega = i(-1)^{n+1}\alpha^\mu_{n+\frac{1}{2}}. \tag{58}$$

c) Our task is to investigate, which states survive the orientifold projection for both the closed and open string sectors. We will make use of a result derived from tadpole cancellation conditions, namely that the closed string vacuum $|0,0,\boldsymbol{p}\rangle$ needs to be orientifold-even whereas the open string vacuum $|0,\boldsymbol{p}\rangle$ may be either even or odd. We proceed by treating each case in turn.

1. As mentioned above, the closed string spectrum contains both left- and right-moving oscillations, travelling in both diretions along the string. The spectrum is balanced by the level-matching condition $N_l = N_r$, which enforces an equal number of left- and right-movers in every physical state. Therefore, raising the vacuum $|0,0,\boldsymbol{p}\rangle$ with $N_l = N_r = 0$ to the first excited level $N_l = N_r = 1$ already requires acting on it with two modes.

An expression capturing all $(D-2)^2 = 576$ linearly independent states with fixed momentum $\boldsymbol{p}$ at first excited level can thus be written as

$$\boldsymbol{\xi}|1,1,\boldsymbol{p}\rangle = \sum_{i,j=1}^{24} \xi_{ij}\,\alpha^i_{l,-1}\alpha^j_{r,-1}|0,0,\boldsymbol{p}\rangle, \tag{59}$$

where ξ_{ij} are the $(D-2)^2$ components of the so-called polarization tensor.

Note: Like any square matrix, ξ_{ij} can be decomposed into a symmetric and antisymmetric part,

$$\xi_{ij} = \frac{1}{2}(\xi_{ij}+\xi_{ji}) + \frac{1}{2}(\xi_{ij}-\xi_{ji}) \equiv S_{ij} + B_{ij}. \tag{60}$$

The symmetric part S_{ij} can be broken down further into a symmetric traceless portion and the trace,

$$S_{ij} = \left(S_{ij} - \frac{\delta^{kl}S_{kl}}{D-2}\,\delta_{ij}\right) + \frac{\delta^{kl}S_{kl}}{D-2}\,\delta_{ij} \equiv g_{ij} + \phi\,\delta_{ij}, \tag{61}$$

where summation over k and l is implied. It is easy to check that g_{ij} is indeed traceless,

$$g^i{}_i = S^i{}_i - \frac{\delta^{kl}S_{kl}}{D-2}\underbrace{\delta^i{}_i}_{D-2} = S^i{}_i - S^l{}_l = 0. \tag{62}$$

All told, the polarization tensor splits into the three parts

$$\xi_{ij} = g_{ij} + B_{ij} + \phi\,\delta_{ij}, \tag{63}$$

each of which is entirely independent of the other two. These three parts have a very interesting physical interpretation. Firstly, note that since the closed string spectrum is organized into levels according to $M^2_{\text{cl}} = \frac{1}{\alpha'}(N_\perp - a_\perp)$, where $a_\perp = 1$ and $N_\perp \equiv N_{l/r}$, all first level states are massless. At this point one might become suspicious when looking at g_{ij}. After all, a massless symmetric tensor field sounds rather reminiscent of an object central to general relativity. Indeed, a closer investigation reveals that g_{ij} describes transversely polarised, spin 2, one-particle states. That is why string theory identifies g_{ij} as the **graviton**. The antisymmetric two-tensor B_{ij} is called the Kalb-Ramond field and can be thought of as a generalised (i.e. higher-rank) **gauge potential**.
Finally, $\phi\,\delta_{ij}$ is just a scaled version of the unit matrix, i.e. has only one degree of freedom. It represents a scalar field and is called the **dilaton**. ϕ will be of crucial importance in the context of string interactions.

Now, going back to our first-excited-level state (59), we calculate its behavior under worldsheet parity.

$$\begin{aligned}\Omega\,\boldsymbol{\xi}|1,1,\boldsymbol{p}\rangle &= \xi_{ij}\,\Omega\,\alpha^i_{l,-1}\underbrace{\Omega^\dagger\Omega}_{\mathbb{1}}\alpha^j_{r,-1}\underbrace{\Omega^\dagger\Omega}_{\mathbb{1}}|0,0,\boldsymbol{p}\rangle \\ &\overset{(49)}{=} \xi_{ij}\,\alpha^i_{r,-1}\alpha^j_{l,-1}|0,0,\boldsymbol{p}\rangle \\ &\overset{i\leftrightarrow j}{=} \xi_{ji}\,\alpha^i_{l,-1}\alpha^j_{r,-1}|0,0,\boldsymbol{p}\rangle = \boldsymbol{\xi}^T|1,1,\boldsymbol{p}\rangle,\end{aligned} \tag{64}$$

where summation over i and j is implied. Also, we used Ω's unitarity $\Omega^\dagger = \Omega^{-1}$, and the fact that it squares to the identity[8] $\Omega^2 = \mathbb{1}$ to write

$$\Omega\,\alpha^i_{l,-1}\Omega^\dagger = (\underbrace{\Omega^\dagger}_{\mathbb{1}})^2\,\Omega\,\alpha^i_{l,-1}\Omega^\dagger\underbrace{\Omega^2}_{\mathbb{1}} = \Omega^\dagger\,\alpha^i_{l,-1}\Omega \overset{(49)}{=} \alpha^i_{r,-1}. \tag{65}$$

So effectively, Ω flips the indices on the polarization tensor $\boldsymbol{\xi}$. This is fine for both g_{ij} and $\phi\,\delta_{ij}$. Since they are symmetric in i and j, the orientifold operator acts like the identity on the

[8]This also implies $(\Omega^\dagger)^2 = \mathbb{1}$ since $(\Omega^\dagger)^2 = (\Omega\,\Omega)^\dagger = \mathbb{1}^\dagger = \mathbb{1}$.

graviton and dilaton. In the case of B_{ij} however, we pick up a sign,

$$\Omega\, B_{ij}\alpha^i_{l,-1}\alpha^j_{r,-1}|0,0,\boldsymbol{p}\rangle = -B_{ij}\alpha^i_{l,-1}\alpha^j_{r,-1}|0,0,\boldsymbol{p}\rangle, \tag{66}$$

which makes the Kalb-Ramond field worldsheet-parity-odd. It therefore falls victim to the orientifold projection.

2. Moving on to the open string sector, we consider strings with Neumann boundary conditions in dimensions X^i, $i \in I$, and ending on a single D-brane, i.e. Dirichlet boundary boundary conditions, in the others, X^j, $j \in J$. We quote our expression from exercise 1 for a general state of this type,

$$(\boldsymbol{A}+\boldsymbol{\phi})|1,\boldsymbol{p}\rangle = \sum_{i\in I} A_i\alpha^i_{-1} + \sum_{j\in J}\phi_j\alpha^j_{-1}|0,\boldsymbol{p}\rangle. \tag{67}$$

Acting with Ω on excitations parallel to the brane gives

$$\begin{aligned}\Omega\,\boldsymbol{A}|1,\boldsymbol{p}\rangle &= A_i\,\underbrace{\Omega\,\alpha^i_{-1}\,\Omega^\dagger}_{(52):\ (-1)^{-1}\alpha^i_{-1}}\,\underbrace{\Omega\,|0,\boldsymbol{p}\rangle}_{\pm|0,\boldsymbol{p}\rangle}\\ &= \mp A_i\alpha^i_{-1}|0,\boldsymbol{p}\rangle = \mp\boldsymbol{A}|1,\boldsymbol{p}\rangle\end{aligned} \tag{68}$$

Apparently, the one-particle photon excitations A_i are projected out of string theory (survive the orientifold action) if the vacuum has a phase of $+1$ (-1).

For excitations ϕ_j that are transverse to the brane, we get

$$\begin{aligned}\Omega\,\boldsymbol{\phi}|1,\boldsymbol{p}\rangle &= \phi_j\,\underbrace{\Omega\,\alpha^j_{-1}\,\Omega^\dagger}_{(55):\ (-1)^{0}\alpha^i_{-1}}\,\underbrace{\Omega\,|0,\boldsymbol{p}\rangle}_{\pm|0,\boldsymbol{p}\rangle}\\ &= \pm\phi_j\alpha^i_{-j}|0,\boldsymbol{p}\rangle = \pm\boldsymbol{\phi}|1,\boldsymbol{p}\rangle,\end{aligned} \tag{69}$$

i.e. it is precisely the other way round. The Goldstone bosons associated with spontaneous breaking of the D-dimensional Poincaré symmetry carry over to (are missing from) the unoriented theory if the vacuum has a phase of $+1$ (-1).

d) For the open strings sector in the presence of N coincident the D-branes, the orientifold action acts on the vacuum

$$\Omega|0,\boldsymbol{p},rs\rangle = \pm|0,\boldsymbol{p},sr\rangle, \tag{70}$$

where the Chan-Paton factors r, s swap since Ω reverses the string.

We express states at the first excited level of this spectrum in terms of a basis of suitable $N\times N$-matrices $\boldsymbol{\lambda}_{rs}$,

$$|n,\boldsymbol{p},\boldsymbol{\lambda}\rangle = \sum_{r,s=1}^{N}\boldsymbol{\lambda}^{rs}\,|n,\boldsymbol{p},r\,s\rangle. \tag{71}$$

These are the same Hermitian Chan-Paton matrices we already encountered in eq. (15) of exercise 1. Looking at eq. (71), it is clear that in order for states to survive the orientifold projection, we need one of the following two combinations of attributes

$$\Omega|0,\boldsymbol{p},rs\rangle = +|0,\boldsymbol{p},sr\rangle \qquad + \qquad \boldsymbol{\lambda}^{rs} = \boldsymbol{\lambda}^{sr}, \tag{72}$$

$$\text{or}\qquad \Omega|0,\boldsymbol{p},rs\rangle = -|0,\boldsymbol{p},sr\rangle \qquad + \qquad \boldsymbol{\lambda}^{rs} = -\boldsymbol{\lambda}^{sr}. \tag{73}$$

So before orientifolding, we are in the oriented theory with N coincident D-branes hosting a $U(N)$ gauge group. The corresponding algebra is spanned by N^2 linearly independent basis states. Of those N^2 states, only $\frac{N}{2}(N+1)$ or $\frac{N}{2}(N-1)$ carry over to the unoriented string theory, depending on whether the vacuum has phase $+1$ or -1, respectively.

As for the change in gauge group precipitated by the orientifold action, we note the following: For strings that are unoriented, the group representations associated with each string end obviously have to be the same. This in particular forces the symmetry group to be one with a *real* fundamental representation, specifically a (special) orthogonal or symplectic group.[9] Since the Lie algebra formed by antisymmetric matrices is related to the Lie group of orthogonal matrices, we formulate the following hypothesis:

$$\text{For a vacuum phase of } \begin{Bmatrix} +1 \\ -1 \end{Bmatrix} \begin{array}{l} \Omega \text{ reduces the gauge group of } N \\ \text{coincident D-branes from } U(N) \text{ to} \end{array} \begin{cases} Sp(N). \\ SO(N). \end{cases} \tag{74}$$

3 Partition function, level density, Hagedorn temperature

In this exercise we will estimate the number of states in the tower of string excitations at a given level. Consider the normalized state (for simplicity, we work in the open NN string sector with light-cone quantization),

$$|n^i_m\rangle \equiv \frac{1}{\sqrt{n!\, m^n}} \, (\alpha^i_{-m})^n \, |0\rangle, \tag{75}$$

where $i \in \{1, \ldots, D-2\}$ enumerates all transverse dimensions in lightcone gauge, i.e. all except $X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^{D-1})$, $m \in \mathbb{N}$ is the oscillation number of modes creating the Fock space from the vacuum, i.e. creators only, and $n \in \mathbb{N}_0$ counts the number of excitations by a certain oscillation number m. Since the $|n^i_m\rangle$ are members of a Fock space, they are also eigenstates of the number operator,

$$\hat{N}_\perp |n^i_m\rangle = m\, n\, |n^i_m\rangle, \qquad \text{where} \quad \hat{N}_\perp = \sum_{m=1}^{\infty} \sum_{i=1}^{24} \alpha^i_{-m} \alpha^i_m. \tag{76}$$

To find the number of states at a given level, we introduce the **partition function**

$$Z(q) = \mathrm{tr}\big(q^{\hat{N}_\perp}\big) \equiv \prod_{m,i} \sum_n \langle n^i_m | q^{\hat{N}_\perp} | n^i_m \rangle. \tag{77}$$

Here $q \in \mathbb{C}$, $|q| < 1$ is some auxiliary variable.

a) Argue that $Z(q)$ is a generating function for the number d_N of states at level N in the sense that

$$Z(q) = \sum_N d_N q^N \tag{78}$$

and that d_N can be computed by the contour integral

$$d_N = \frac{1}{2\pi i} \int\limits_{|q|=1} \mathrm{d}q \, q^{-N-1} Z(q). \tag{79}$$

b) Show that

$$Z(q) = \prod_{m=1}^{\infty} (1 - q^m)^{-24}. \tag{80}$$

Hint: You may have encountered a similar computation in statistical mechanics.

Remark: $Z(q)$ can be rewritten as

$$Z(q) = q^2 \eta(\tau)^{-24}, \qquad \eta(\tau) = \exp\left(\frac{i\pi\tau}{12}\right) \prod_{n=1}^{\infty} (1 - e^{2\pi i n \tau}), \qquad q = e^{2\pi i \tau}, \qquad \tau \in \mathbb{C}, \tag{81}$$

in terms of the famous **Dedekind eta-function**.

[9]Symplectic matrices are automatically orientation preserving, i.e. have a determinant of one. Orthogonal matrices can either preserve or reverse orientation, so we really need the special orthogonal matrices.

c) Now parametrise $q = \rho\, e^{i\phi}$ so that

$$\mathrm{d}q = e^{i\phi}\,\mathrm{d}\rho + i\rho e^{i\phi}\,\mathrm{d}\phi = \frac{q}{\rho}\,\mathrm{d}\rho + iq\,\mathrm{d}\phi, \tag{82}$$

and pick a contour with constant ρ so that $\int\limits_{|q|=\rho} \mathrm{d}q = i\int_{-\pi}^{\pi} q\,\mathrm{d}\phi$ to argue that

$$d_N = \frac{1}{2\pi}\int_{-\pi}^{\pi} q e^{f(q)}\,\mathrm{d}\phi, \qquad f(q) = -(N+1)\ln(q) - 24\sum_{m=1}^{\infty}\ln(1-q^m). \tag{83}$$

We will now crudely approximate d_N by the so-called **saddle-point approximation**: Suppose there is some $q_0 = \rho_0\, e^{i\phi_0}$ such that $f(q)$ has a sharply peaked maximum in the ϕ_0-direction at q_0. Suppose furthermore that the prime contribution to the integral comes from the region around that maximum, i.e. the integral localises around this maximum/saddle point. Under these assumptions argue that we can approximate

$$d_N \approx \frac{1}{2\pi} q_0 e^{f(q_0)}\int_{-\pi}^{\pi}\mathrm{d}\phi\,\exp\left(\frac{1}{2}\frac{\partial^2 f}{\partial^2\phi}\bigg|_{q_0}\phi^2 + i\phi\right), \tag{84}$$

and approximate further

$$d_N \approx \frac{1}{\sqrt{2\pi}}\frac{e^{f(q_0)}}{\sqrt{\frac{\partial^2 f}{\partial^2 q}\Big|_{q_0}}}\exp\left[-\left(2q_0^2\,\frac{\partial^2 f}{\partial^2 q}\bigg|_{q_0}\right)^{-1}\right]. \tag{85}$$

d) It turns out that for $N \gg 1$ the function $f(q)$ has such a sharp maximum at $q_0 \approx 1 - \frac{2\pi}{\sqrt{N}} = 1 - x$ (i.e. $x \ll 1$). The simplest way to extract the very crude behaviour of d_N is to approximate

$$f(q) \approx \frac{4\pi^2}{1-q} - -(N+1)\ln(q), \tag{86}$$

for $q = 1 - x$, $x \ll 1$. Show this using the series

$$\ln(1-q) = -\sum_{m=1}^{\infty}\frac{q^m}{m}, \qquad \sum_{m=1}^{\infty}\frac{1}{m^2} = \frac{\pi^2}{6}. \tag{87}$$

Finally deduce that in this primitive approximation

$$d_N \approx N^{-\frac{3}{4}} e^{4\pi\sqrt{N}}. \tag{88}$$

Remark: A more more accurate approximation using non-trivial properties of the Dedekind function yields

$$d_N \approx N^{-\frac{27}{4}} e^{4\pi\sqrt{N}}, \tag{89}$$

which is indeed the correct leading term in the famous Hardy-Ramanujan formula for the asymptotic behaviour of the number of partitions of a large integer N.

e) Use the relation $\alpha' M^2 = N - 1$, the macroscopic definition of entropy and the thermodynamic relation between entropy and temperature to argue that for high energies there exists a maximum temperature, the Hagedorn temperature T_{H}, of the order

$$k_B T_{\mathrm{H}} \approx \frac{1}{4\pi\sqrt{\alpha'}}. \tag{90}$$

Note: The appearance of such a maximum temperature might play an important role in early cosmology and is a consequence of the exponential degeneracy of the string tower at high energy levels.

a) First, we show that $Z(q) \equiv \sum_{m,n,i} \langle n^i_m | q^{\hat{N}_\perp} | n^i_m \rangle$ is equal to $\sum_N d_N q^N$. By Taylor expanding $q^{\hat{N}_\perp}$,

$$q^{\hat{N}_\perp} = e^{\ln(q)\,\hat{N}_\perp} = 1 + \ln(q)\hat{N}_\perp + \frac{1}{2}\ln(q)^2\hat{N}_\perp^2 + \frac{1}{3!}\ln(q)^3\hat{N}_\perp^3 + \dots \tag{91}$$

and inserting into the matrix element in the partition function, we get

$$\begin{aligned}\langle n^i_m | q^{\hat{N}_\perp} | n^i_m \rangle &= \langle n^i_m | 1 | n^i_m \rangle + \langle n^i_m | \ln(q)\hat{N}_\perp | n^i_m \rangle + \frac{1}{2}\langle n^i_m | \ln(q)^2 \hat{N}_\perp^2 | n^i_m \rangle + \dots \\ &\overset{10}{=} 1 + \ln(q)\, m\, n + \frac{1}{2}\ln(q)^2 (m\,n)^2 + \dots \\ &= q^{m\,n},\end{aligned} \tag{92}$$

and hence $Z(q) = \prod_{m,i} \sum_n q^{m\,n}$. This expression is already promisingly close to the generating function of d_N that we are after. What we need to do now is count how often q appears in the sum with different combinations of m and n that multiply to give the same exponent N. To that end, note that since $N = m\,n$ is the eigenvalue of the number operator, and it ended up in the exponent of q by summing over all states in the Fock space, it is clear that q^N appears exactly as often as the eigenvalue N in the spectrum, i.e. with the degeneracy d_N of states at level N. Therefore,

$$Z(q) = \prod_{m,i} \sum_n q^{m\,n} = \sum_{N=0}^{\infty} d_N\, q^N. \tag{93}$$

Note: Out of interest, we want to do some state counting and find out what values d_N actually takes.

- Start with the case $N = 0$, i.e. $\hat{N}_\perp | n^i_m \rangle = 0$. There is only one state that satisfies this equation and that is the vacuum $|0\rangle$. Hence, $d_0 = 1$.
- To get $\hat{N}_\perp | n^i_m \rangle = 1 | n^i_m \rangle$, we need to excite the vacuum exactly once, i.e. $|1^i_1\rangle = \alpha^i_{-1}|0\rangle$. But it doesn't matter which of the $(D-2)$ transverse dimensions we choose to excite, i.e. i is random here but summed over in the number operator (and the partition function), so we get a degeneracy of $d_1 = (D-2) = 24$ at the first excited level.
- For $\hat{N}_\perp | n^i_m \rangle = 2 | n^i_m \rangle$, there are now two possibilities,

$$|1^i_2\rangle = \alpha^i_{-2}|0\rangle, \tag{94}$$

$$|1^i_1; 1^j_1\rangle = |1^i_1\rangle \otimes |1^j_1\rangle = \alpha^i_{-1}\alpha^j_{-1}|0\rangle. \tag{95}$$

  $|1^i_2\rangle$ again amounts to a vector of 24 degenerate states. $|2^i_1 j\rangle$ can by thought of as a $24 \times 24 = 576$-dimensional matrix. Note, however, that it doesn't matter here which dimension, i or j, we excite first. The resulting state will be the same either way. We are therefore dealing with a symmetric matrix with only $\frac{24}{2}(24+1) = 300$ independent entries. So the second excited level has a degeneracy of $d_2 = 24 + 300 = 324$.
- A state with $\hat{N}_\perp | n^i_m \rangle = 3 | n^i_m \rangle$ can be constructed by three distinct mode combinations,

$$|1^i_3\rangle = \alpha^i_{-3}|0\rangle, \tag{96}$$

$$|2^i_1; 1^j_1\rangle = |2^i_1\rangle \otimes 1^j_1\rangle = \alpha^i_{-2}\alpha^j_{-1}|0\rangle, \tag{97}$$

$$|1^i_1; 1^j_1; 1^k_1\rangle = |1^i_1\rangle \otimes 1^j_1\rangle \otimes 1^k_1\rangle = \alpha^i_{-1}\alpha^j_{-1}\alpha^k_{-1}|0\rangle. \tag{98}$$

  $|1^i_3\rangle$ gives 24 states. Since the oscillation number of the modes creating $|2^i_1; 1^j_1\rangle$ now differ, this matrix is not symmetric and we get the full $(D-2)^2 = 576$ states. Finally, $|1^i_1; 1^j_1; 1^k_1\rangle$ is a three dimensional matrix that is symmetric in all three indices. It therefore gives another $\frac{24}{6}(24+1)(24+2) = 2600$ states. All told, the degeneracy of level 3 is $d_3 = 24 + 576 + 2600 = 3200$.

[10] We used here that the states are normalized to one, $\langle n^i_m | n^i_m \rangle = 1$.

We could continue indefinitely with this exercise, but it grows increasingly tedious the larger N becomes. In any case, we already have enough information at this point to recognize a pattern. The numbers 1, 24, 324, and 3200 are precisely the first four coefficients in the expansion of $\frac{e^{2\pi i\tau}}{\eta^{24}(\tau)}$, where $\eta(\tau)$ is number theory's Dedekind eta-function,

$$\frac{e^{2\pi i\tau}}{\eta^{24}(\tau)} = 1 + 24\,e^{2\pi i\tau} + 324\,e^{4\pi i\tau} + 3200\,e^{6\pi i\tau} + 25650\,e^{8\pi i\tau} + \dots \tag{99}$$

Now that we have verified eq. (78), it is a simple matter to also check eq. (79).

$$\begin{aligned} \frac{1}{2\pi i}\int\limits_{|q|=1} \mathrm{d}q\, q^{-N-1} Z(q) &\overset{(78)}{=} \frac{1}{2\pi i}\int\limits_{|q|=1} \mathrm{d}q\, q^{-N-1} \sum_{N'=0}^{\infty} d_{N'}\, q^{N'} \\ &= \frac{1}{2\pi i}\sum_{N'=0}^{\infty} d_{N'} \overbrace{\int\limits_{|q|=1} \mathrm{d}q\, q^{N'-N-1}}^{2\pi i\,\delta_{N'-N-1,-1},\text{ since } |q|<1} = \sum_{N'=0}^{\infty} d_{N'}\,\delta_{N',N} = d_N. \end{aligned} \tag{100}$$

b) Next we show eq. (80). Finally, an exercise that requires nothing but some honest calculations.

$$Z(q) \overset{(77)}{=} \prod_{m,i}\sum_{n} \langle n^i_m | q^{\hat{N}_\perp} | n^i_m\rangle \overset{(92)}{=} \prod_{m,i}\sum_{n} q^{m\,n} = \prod_{m,i}\frac{1}{1-q^m} = \prod_{m=1}^{\infty}(1-q^m)^{-24}. \tag{101}$$

c) To prepare the contour integral representation (79) of d_N for a saddle-point approximation, we parametrize the auxiliary variable as $q = \rho e^{i\phi}$ so that the contour integral for some constant $\rho \neq 0$ becomes $\int\limits_{|q|=\rho} \mathrm{d}q \overset{(82)}{=} i\int_{-\pi}^{\pi} q\,\mathrm{d}\phi$.

$$\begin{aligned} d_N &= \frac{1}{2\pi i}\int\limits_{|q|=\rho} \mathrm{d}q\, q^{-N-1} Z(q) \overset{(101)}{=} \frac{i}{2\pi i}\int_{-\pi}^{\pi} q\,\mathrm{d}\phi\, e^{-(N+1)\ln(q)} \prod_{m=1}^{\infty}(1-q^m)^{-24} \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} q\,\mathrm{d}\phi\, \exp\Big(-(N+1)\ln(q) - 24\sum_{m=1}^{\infty}\ln(1-q^m)\Big) \equiv \frac{1}{2\pi}\int_{-\pi}^{\pi} q\,\mathrm{d}\phi\, e^{f(q)}, \end{aligned} \tag{102}$$

where $f(q) = -(N+1)\ln(q) - 24\sum_{m=1}^{\infty}\ln(1-q^m)$.

Now for the actual saddle-point approximation: We assume there exists a $q_0 = \rho_0\, e^{i\phi_0}$ such that $f(q)$ has a sharp maximum at ρ_0 in ϕ_0-direction and lay our ϕ-contour at a distance of ρ_0 from the origin so that we integrate right through this maximum. We further assume that said maximum lies on the real axis, i.e. $\phi_0 = 0$ (see part d) of the exercise for confirmation). Then the Taylor expansion of $f(q)$ around $\phi = 0$ with ρ held constant at ρ_0 reads

$$f(q)\big|_{\rho_0} = \underbrace{f(\rho,\phi)\big|_{\rho_0,\phi_0}}_{f(q_0)} + \underbrace{\frac{\partial f(\rho,\phi)}{\partial\phi}\bigg|_{\rho_0,\phi_0}}_{0\text{ at maximum}}\phi + \frac{1}{2}\frac{\partial^2 f(\rho,\phi)}{\partial^2\phi}\bigg|_{\rho_0,\phi_0}\phi^2 + \mathcal{O}[\phi^3], \tag{103}$$

By Inserting eq. (103) into eq. (102) and neglecting terms cubic and higher in q, we get

$$d_N \approx \frac{1}{2\pi}\int_{-\pi}^{\pi} q\,\mathrm{d}\phi\, e^{f(q_0) + \frac{1}{2}\frac{\partial^2 f(\rho,\phi)}{\partial^2\phi}\big|_{q_0}\phi^2} = \frac{q_0}{2\pi} e^{f(q_0)} \int_{-\pi}^{\pi}\mathrm{d}\phi\, e^{\frac{1}{2}\frac{\partial^2 f(\rho,\phi)}{\partial^2\phi}\big|_{q_0}\phi^2 + i\phi}, \tag{104}$$

where we used that due to integrating along a circle with radius ρ_0, we have $q = \rho_0 e^{i\phi} = q_0 e^{i\phi}$.

At this point, a third approximation becomes necessary. Due to $f(q)$'s sharp maximum at q_0, the integral (104) localizes around ϕ_0. This allows us to extend the integration bounds without much affecting the result, so we may as well extend them all the way out to $\pm\infty$ and solve it as a Gaussian integral using the well-known formula $\int_{-\infty}^{\infty} \exp(-ax^2+bx+c)\,\mathrm{d}x = \sqrt{\pi/a}\,\exp(\frac{b^2}{4a}+c)$.

$$d_N \approx \frac{q_0}{2\pi} e^{f(q_0)} \sqrt{\frac{\pi}{-\frac{1}{2}\left.\frac{\partial^2 f(\rho,\phi)}{\partial^2\phi}\right|_{q_0}}} \exp\left(\frac{i^2}{-\frac{4}{2}\left.\frac{\partial^2 f(\rho,\phi)}{\partial^2\phi}\right|_{q_0}}\right) = \frac{1}{\sqrt{2\pi}} \frac{e^{f(q_0)}}{\sqrt{\left.\frac{\partial^2 f(q)}{\partial^2 q}\right|_{q_0}}} \exp\left(-\frac{1}{2q_0^2 \left.\frac{\partial^2 f(q)}{\partial^2 q}\right|_{q_0}}\right), \tag{105}$$

where in the last step we used $\partial^2 q|_{q_0} = -q_0^2 \partial^2\phi|_{q_0}$.

d) The series in $f(q) = -(N+1)\ln(q) - 24\sum_{m=1}^{\infty}\ln(1-q^m)$ can be **crudely** approximated as

$$-24\sum_{m=1}^{\infty}\ln(1-q^m) = 24\sum_{m=1}^{\infty}\sum_{k=1}^{\infty}\frac{(q^m)^k}{k} \approx 24\sum_{m=0}^{\infty}\sum_{k=1}^{\infty}\frac{q^m}{k^2} = \frac{24\pi^2}{6}\sum_{m=0}^{\infty} q^m = \frac{4\pi^2}{1-q}, \tag{106}$$

where we used

$$\ln(1-q^m) = -\sum_{k=1}^{\infty}\frac{(q^m)^k}{k}, \qquad \text{and} \qquad \sum_{k=1}^{\infty}\frac{1}{k^2} = \frac{\pi^2}{6}, \tag{107}$$

in the first and third step, respectively.

The exercise asserts that for large N, $f(q)$ does indeed have a sharp maximum on the real line approximately at $q_0 \approx 1 - \frac{2\pi}{\sqrt{N}}$. Using this information and our approximate $f(q)$ with eq. (106) inserted, we calculate

$$e^{f(q_0)} \approx e^{\frac{4\pi^2}{1-q_0} - (N+1)\ln(q_0)} \overset{q_0 \approx 1-\frac{2\pi}{\sqrt{N}}}{\approx} e^{2\pi\sqrt{N}}\left(1-\frac{2\pi}{\sqrt{N}}\right)^{-(N+1)} \approx e^{4\pi\sqrt{N}}, \tag{108}$$

where we used that the series expansion of $\left(1-\frac{2\pi}{\sqrt{N}}\right)^{-(N+1)}$ at $N\to\infty$ is

$$\left(1-\frac{2\pi}{\sqrt{N}}\right)^{-(N+1)} = \exp\left(2\pi\sqrt{N} + 2\pi^2 + \frac{2\pi}{\sqrt{N}} + \ldots\right). \tag{109}$$

Next, we compute

$$\begin{aligned} \left.\frac{\partial^2 f(q)}{\partial^2 q}\right|_{q_0} &\approx \frac{\partial}{\partial q}\left[\frac{4\pi^2}{(1-q)^2} - \frac{N+1}{q}\right]\Bigg|_{q_0} = \left.\frac{8\pi^2}{(1-q)^3} + \frac{N+1}{q^2}\right|_{q_0} \\ &\approx \frac{8\pi^2}{\frac{8\pi^3}{N^{\frac{3}{2}}}} + \frac{N+1}{q_0^2} = \frac{N^{\frac{3}{2}}}{\pi} + \frac{N+1}{q_0^2} \xrightarrow{N\to\infty} \frac{N^{\frac{3}{2}}}{\pi}, \end{aligned} \tag{110}$$

in order to see that

$$\exp\left(-\frac{1}{2q_0^2\left.\frac{\partial^2 f(q)}{\partial^2 q}\right|_{q_0}}\right) \approx \exp\left(-\frac{1}{2q_0^2\frac{N^{\frac{3}{2}}}{\pi} + 2(N+1)}\right) \xrightarrow{N\to\infty} 1. \tag{111}$$

Putting everything back into eq. (105), we obtain the heavily approximated result for the large-N behavior of d_N,

$$\lim_{N\to\infty} d_N \approx \frac{1}{\sqrt{2\pi}}\frac{e^{4\pi\sqrt{N}}}{\sqrt{\frac{N^{\frac{3}{2}}}{\pi}}} \approx N^{-\frac{3}{4}} e^{4\pi\sqrt{N}}. \tag{112}$$

Considering the large number of daredevil approximations we performed, this result is still remarkably close to $d_N \approx N^{-\frac{27}{4}} e^{4\pi\sqrt{N}}$, which can be derived by leveraging more involved number-theoretic machinery.

e) A system's macroscopic entropy S is given by the logarithm of the total number of microstates Ω the system can occupy. That number obviously depends on the system's energy E, or in the case of a string, on the mass M of its state. Fortunately, all we've been doing in this exercise is to derive a relation d_N between the mass level N of a state and its degeneracy with respect to the number operator $\hat{N}_\perp$, so we can directly insert d_N for Ω to get

$$S(N) = \ln(d_N) = \ln\left(N^{-\frac{3}{4}} e^{4\pi\sqrt{N}}\right) = 4\pi\sqrt{N} + \ln(N^{-\frac{3}{4}}) \xrightarrow{N\to\infty} 4\pi\sqrt{N}. \tag{113}$$

The relation $\alpha' M^2 = N - a$ gives all masses in the tower of states. For highly excited states with large N, we can neglect the normal ordering constant $a \leq 1$, i.e. $\sqrt{\alpha'}M = \sqrt{N}$. Thus the entropy as a function of mass for large M reads

$$S(M) \approx 4\pi\sqrt{\alpha'}M. \tag{114}$$

The macroscopic temperature can be calculated from $S(M)$ by simple differentiation,

$$T = \left(\frac{\partial S(M)}{\partial M}\right)^{-1} = \frac{1}{4\pi\sqrt{\alpha'}} \equiv T_{\mathrm{H}}. \tag{115}$$

Interestingly, the temperature of strings with high mass, i.e. energy, is entirely constant. We could increase the energy to arbitrarily high levels and yet never break this temperature cutoff called the Hagedorn temperature.