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Quantum Field Theory II - Assignment 4

Problem 4.1 (The 1PI effective action)

In the lecture, we have defined

$$\tau(x_1,\dots,x_n) = \frac{\delta}{i\delta J(x_1)} \cdots \frac{\delta}{i\delta J(x_n)}\, iW[J], \qquad (1)$$

$$\tilde{\Gamma}(x_1,\dots,x_n) = \frac{\delta}{\delta\varphi(x_1)} \cdots \frac{\delta}{\delta\varphi(x_n)}\, i\Gamma[\varphi]. \qquad (2)$$

What is the interpretation of $\tau(x_1,\dots,x_n)$ and $\Gamma(x_1,\dots,x_n)$?
We have presented both a computational and a graphic method to relate $\tau(x_1,x_2,x_3)$ and $\Gamma(x_1,x_2,x_3)$. Revise these methods and use them to relate $\tau(x_1,x_2,x_3,x_4)$ and $\Gamma(x_1,x_2,x_3,x_4)$. Argue that the result demonstrates that $\Gamma[\varphi]$ is indeed a generating functional of the 1PI - correlators.

In analogy to $G_J(x_1,\dots,x_n) = \prod_{j=1}^{n} \frac{\delta}{i\delta J(x_j)} \frac{Z[J]}{Z[0]}$, where $\frac{Z[J]}{Z[0]}$ is the generating functional for all n-point Green's functions $G_J(x_1,\dots,x_n)$ in the presence of the source J, $\tau(x_1,\dots,x_n) \equiv G_J^{(c)}(x_1,\dots,x_n) = \prod_{j=1}^{n} \frac{\delta}{i\delta J(x_j)}\, iW[J]$ denotes a fully connected n-point Green's function in the presence of J and $iW[J]$ is called the effective action. The fully connected are a subset of all Green's functions. An even stronger reduction are the fully connected, amputated 1PI Green's functions $\tilde{\Gamma}(x_1,\dots,x_n)$ in the presence of the field φ (or without $\Gamma(x_1,\dots,x_n) = \tilde{\Gamma}(x_1,\dots,x_n)|_{\varphi=0}$) generated by the 1PI effective action $\Gamma[\varphi]$.

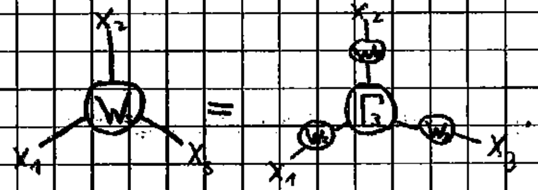
In the case of $n=3$, $\tau(x_1,x_2,x_3)$ and $\tilde{\Gamma}(y_1,y_2,y_3)$ are related as follows,

$$\tau(x_1,x_2,x_3) = \int d^4y_1\, d^4y_2\, d^4y_3\; \tau(x_1,y_1)\,\tau(x_2,y_2)\,\tau(x_3,y_3)\,\tilde{\Gamma}(y_1,y_2,y_3)$$

or displayed graphically

$\tau_3(x_1,x_2,x_3) = (W)|_3 =$ 

or illustrated in diagrams,



For $n=4$, we need to consider

$$\tau(x_1,x_2,x_3,x_4) = \frac{\delta}{i\delta J(x_4)}\,\tau(x_1,x_2,x_3) = \frac{\delta}{i\delta J(x_4)}\int d^4y_1\, d^4y_2\, d^4y_3\, \tau(x_1,y_1)\,\tau(x_2,y_2)\,\tau(x_3,y_3)\,\tilde{\Gamma}(y_1,y_2,y_3)$$

By means of the chain rule we can rewrite the functional derivative w.r.t. $iJ(x_4)$ as one w.r.t. $\varphi(y_4)$,

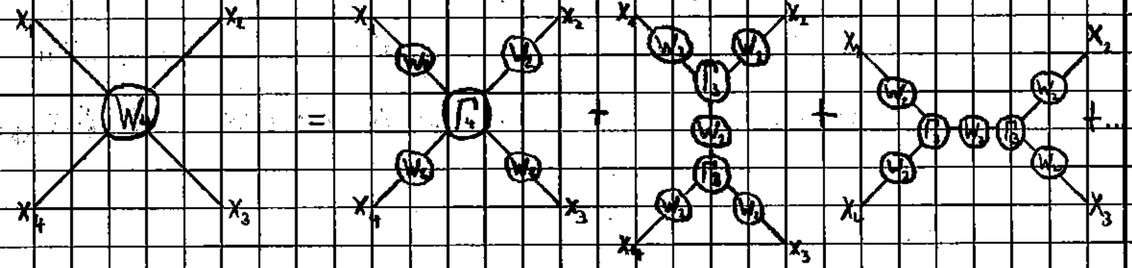
$$\frac{\delta}{i\delta J(x_4)} = \int d^4y_4\, \frac{\delta\overbrace{\varphi(y_4)}^{=\frac{\delta W[J]}{i\delta J(y_4)}}}{i\delta J(x_4)}\,\frac{\delta}{\delta\varphi(y_4)} = \int d^4y\, \underbrace{\frac{\delta^2 W[J]}{i\delta J(x_4)\, i\delta J(y_4)}}_{\tau_2(x_4,y_4)}\,\frac{\delta}{\delta\varphi(y_4)} = \int d^4x\, \tau_2(x_4,y_4)\,\frac{\delta}{\delta\varphi(y_4)}.$$

Applying the product rule, we get four terms:

$$\tau(x_1,x_2,x_3,x_4) = \int\prod_{k=1}^{4} d^4y_k\, \tau(x_k,y_k)\underbrace{\frac{\delta}{\delta\varphi(y_4)}\tilde{\Gamma}(y_1,y_2,y_3)}_{\tilde{\Gamma}(y_1,y_2,y_3,y_4)} + \int\prod_{k=1}^{4} d^4y_k\, \tau(x_1,y_1,y_4)\,\tau(x_2,y_2)\,\tau(x_3,y_3)\,\tilde{\Gamma}(y_1,y_2,y_3)$$

$$+\int\prod_{k=1}^{4} d^4y_k\, \tau(x_1,y_1)\,\tau(x_2,y_2,y_4)\,\tau(x_3,y_3)\,\tilde{\Gamma}(y_1,y_2,y_3) + \int\prod_{k=1}^{4} d^4y_k\, \tau(x_1,y_1)\,\tau(x_2,y_2)\,\tau(x_3,y_3,y_4)\,\tilde{\Gamma}(y_1,y_2,y_3)$$

To make better sense of the above, we may insert the expression for the case $n=3$, i.e. $\tau(x_1,x_2,x_3)$. Illustrated again in terms of diagrams, we found that



This result was to be expected since it confirms for the case $n=4$ that $\tilde{\Gamma}_4$ gives indeed the amputated, fully connected 1PI 4-point Green's function. The legs merely carry corrections W_2 due to the fully (Dyson) resummed Propagator $G_2 =$  $\big|_{J=0} =$ , and the tree additional terms are one-particle reducible (1PR) diagrams that can be split with a single cut at the central W_2.

Therefore, the fourth derivative of $i\Gamma[\varphi]$ w.r.t. φ is exactly the 1PI 4-point function.

Problem 4.2 (Coleman-Weinberg potential for ϕ^4-theory)

In this exercise, we calculate the quantum effective action $\Gamma[\varphi]$ for the ϕ^4-theory,

$$S[\phi] = \int d^4x \left(-\frac{1}{2}\phi(x)(\partial_x^2 + \tilde{m}^2)\phi(x) - \frac{\lambda}{4!}\phi^4(x) \right), \quad \tilde{m}^2 = m^2 - i\epsilon \qquad (3)$$

up to one-loop order for a constant background field $\varphi = \varphi_0$. In the lecture, it was shown that

$$\Gamma[\varphi] = S[\varphi] + \hbar \, \gamma^{1\text{-loop}}[\varphi] + \text{'higher loop corrections'}, \text{ with} \qquad (4)$$

$$\gamma^{1\text{-loop}}[\varphi] = -i \ln \int \hat{D}f \, e^{-\frac{1}{2} f \cdot \left(-\frac{i}{\hbar} \frac{\delta^2 S[\varphi]}{\delta\varphi\delta\varphi} \right) \cdot f}, \qquad (5)$$

where the measure $\hat{D}$ is defined such that $Z[0] = 1$.

a) Show that for any Hermitian matrix A, $\ln(\det A) = \operatorname{Tr}(\ln A)$.

Hint: You may find it useful to perform a basis change for which A becomes diagonal.

Any complex normal matrix (satisfying $A^\dagger A = AA^\dagger$) has an orthogonal eigenvector basis. Assembling these eigenvectors into the unitary matrix U ($UU^\dagger = U^\dagger U = \mathbb{1}$), any complex normal matrix A can be decomposed

$$A = UDU^\dagger,$$

where D is a diagonal matrix which, if A is also hermitian ($A = A^\dagger$), has only real values, namely the eigenvalues λ_i of A.

Performing such a change of basis, we get

$$\ln(\det A) = \ln(\det(UDU^\dagger)) = \ln(\det U \cdot \det D \cdot \det U^\dagger)$$
$$= \ln\Big(\det(\underbrace{UU^\dagger}_{\mathbb{1}}) \prod_i \lambda_i\Big) = \sum_i \ln(\lambda_i) = \operatorname{Tr}(\ln A)$$

b) Evaluate $K^{1\text{-loop}}$ by means of the path integral techniques presented in assignments 2 and 3. In analogy to part a), use that as for a differential operator D, $\ln(\det D) = \mathrm{Tr}(\ln D)$. How do you interpret the trace?

Hint: The result can be found in the lecture notes.

$$K^{1\text{-loop}}[\varphi] = -i \ln \int \tilde{D}\phi \; e^{-\frac{1}{2} f \cdot \left(\frac{\delta [S[\varphi]]}{\hbar\, \delta\phi\delta\phi} \right) \cdot f},$$

where $\int \tilde{D}\phi := \frac{1}{Z[0]} \int D\phi = \frac{1}{Z_0[0] + \sigma(\phi^4)} \int D\phi$. Since we are performing here a perturbative expansion in powers of $\hbar$, where $K^{1\text{-loop}}[\varphi]$ corresponds to order $\hbar^1$ whereas $\sigma(\phi^4)$ contains higher powers of $\hbar$, we drop the latter and only carry along for normalisation the generating functional $Z_0[0]$, of the free theory at zero source J, given by

$$Z_0[0] = \int D\phi\, e^{iS_0[\varphi]}, \qquad \text{with } S_0[\varphi] = S[\varphi]\big|_{\lambda=0}.$$

Before performing the path integral, we explicitly calculate the exponent

$$\frac{-i}{2\hbar} \frac{\delta^2 S[\varphi]}{\delta\varphi(x)\,\delta\varphi(y)} = \frac{-i}{2\hbar} \underbrace{\frac{\delta}{\delta\varphi(x)} \frac{\delta}{\delta\varphi(y)} \int d^4x' \left(-\frac{1}{2} \phi(x') \left(\partial_{x'}^2 + \hat{m}^2 \right) \phi(x') - \frac{\lambda}{4!} \phi^4(x') \right)}_{-(\partial_y^2 + \hat{m}^2)\,\delta(x-y)^{(4)},\ \text{see eq. (5), assignment 2}}$$

$$= \frac{i}{2\hbar} \left[(\partial_x^2 + \hat{m}^2)\, \delta^{(4)}(x-y) + \frac{\lambda}{2} \phi^2(x)\, \delta^{(4)}(x-y) \right] =: D.$$

Now, inserting D back into our expression for $K^{1\text{-loop}}[\varphi]$ and using our result from exercise 3.2 d), more specifically

$$\int D\phi\, e^{-\phi \cdot D \cdot \phi} = \sqrt{\frac{\pi^n}{\det D}}, \quad \text{for some differential operator } D,$$

where n is simply the dimensionality of the integral measure, we can carry out the path integration for a static background field $\varphi = \varphi_0$ to arrive at

$$K^{1\text{-loop}}[\varphi_0] = -i \ln \frac{\int D\phi\, e^{-\phi \cdot D \cdot \phi}}{\int D\phi\, e^{-\phi \cdot D_{\lambda=0} \cdot \phi}} \Bigg|_{\phi=\phi_0} = -i \ln \frac{\sqrt{\frac{\pi^n}{\det D}}}{\sqrt{\frac{\pi^n}{\det D_{\lambda=0}}}} \Bigg|_{\phi=\phi_0} = -\frac{i}{2} \ln \left(\frac{\det D_{\lambda=0}}{\det D} \right) \Bigg|_{\phi=\phi_0}$$

09.05.2015

Using now the expression we showed in part a), we can simplify further

$$K^{1\text{-loop}}[\phi_0] = -\frac{i}{2}\ln\left(\frac{\det D_{\lambda=0}}{\det D}\right)\bigg|_{\phi=\phi_0} = -\frac{i}{2}\left(\ln\left(\det D_{\lambda=0}\right) - \ln\left(\det D_{\phi=\phi_0}\right)\right)$$

$$= -\frac{i}{2}\left(\mathrm{Tr}\left(\ln D_{\lambda=0}\right) - \mathrm{Tr}\left(\ln D_{\phi=\phi_0}\right)\right) = -\frac{i}{2}\,\mathrm{Tr}\left(\ln\left(\frac{D_{\lambda=0}}{D_{\phi=\phi_0}}\right)\right)$$

$$= -\frac{i}{2}\,\mathrm{Tr}\left(\ln\left(\frac{\frac{i}{2\pi}(\partial_x^2+\hat{m}^2)\delta(x-y)}{\frac{i}{2\pi}(\partial_x^2+\hat{m}^2+\frac{\lambda}{2}\phi_0^2)\delta(x-y)}\right)\right) = -\frac{i}{2}\,\mathrm{Tr}\left(\ln\left(\frac{\partial_x^2+\hat{m}^2}{\partial_x^2+\hat{m}^2+\frac{\lambda}{2}\phi_0^2}\right)\right)$$

The trace of a differential operator is given by the sum of its eigenvalues, for example a common operator trace appearing in statistical physics is $Z = \mathrm{Tr}\, e^{-\beta H} = \sum_n \langle n|e^{-\beta H}|n\rangle = \sum_n \langle n|e^{-\beta E_n}|n\rangle = \sum_n e^{-\beta E_n}$.

A suitable eigenbasis for ∂_x^2 is e^{ikx}, however since D's eigenvalues are continuous rather than discrete, we need to integrate over the basis

$$K^{1\text{-loop}}[\phi_0] = \frac{i}{2}\mathrm{Tr}\left(\ln\left(\frac{\partial_x^2+\hat{m}^2+\frac{\lambda}{2}\phi_0^2}{\partial_x^2+\hat{m}^2}\right)\right) := \int d^4x\int d^4k\, \underbrace{\langle k|x\rangle}_{\frac{1}{(2\pi)^2}e^{ikx}}\, \frac{i}{2}\ln\left(\frac{\partial_x^2+\hat{m}^2+\frac{\lambda}{2}\phi_0^2}{\partial_x^2+\hat{m}^2}\right)\underbrace{\langle x|k\rangle}_{\frac{1}{(2\pi)^2}e^{-ikx}}$$

$$= \frac{i}{2}\int d^4x\int\frac{d^4k}{(2\pi)^4}\ln\left(\frac{-k^2+\hat{m}^2+\frac{\lambda}{2}\phi_0^2}{-k^2+\hat{m}^2}\right)e^{i(kx-kx)} = \frac{i}{2}V_{\mathbb{R}^{1,3}}\int\frac{d^4k}{(2\pi)^4}\ln\left(\frac{\hat{m}^2+\frac{\lambda}{2}\phi_0^2-k^2}{\hat{m}^2-k^2}\right)$$

where we used the Taylor series representation of the logarithm,

$$\ln\left(\partial_x^2\right) = -\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(-1+\partial_x^2\right)^n,$$

to let ∂_x^2 act directly on its eigenstates.

c) We define the effective potential $V[\varphi] = -\mathrm{Vol}_{\mathbb{R}^{1,3}}\,\Gamma[\varphi]$. Give the potential up to one loop-order, $V^{\text{tree}}(\varphi=\varphi_0) + V^{1\text{-loop}}(\varphi=\varphi_0)$.

Hint: Starting from the expression found in part b), you should find an integral repesentation of $V^{1\text{-loop}}(\varphi=\varphi_0)$ of the form

$$\int d^3k \left(\sqrt{\vec{k}^2 + X + \delta} - \sqrt{\vec{k}^2 + X}\right). \qquad (6)$$

To this end, perform the integral over k^0 as a suitable contour integral. Furthermore, the following integral identity might be useful,

$$\int_X^{X+\delta} d\xi\, \frac{1}{\xi} = \ln\left(\frac{X+\delta}{X}\right).$$

Give a physical interpretation of the two terms in eq. (6).

Considering definitions given in the script, i.e. eqs. (7.198)-(7.202), we deduce the following:

$$\Gamma[\varphi] := S[\varphi] + \hbar K[\varphi], \qquad \text{with } K[\varphi] = K^{1\text{-loop}}[\varphi] + \mathcal{O}(\hbar^2), \quad \text{and}$$

$$\Gamma[\varphi] = -\mathrm{Vol}_{\mathbb{R}^{1,3}}\, V(\varphi) = -\mathrm{Vol}_{\mathbb{R}^{1,3}}\left(V^0(\varphi) + V^{1\text{-loop}}(\varphi) + \dots\right)$$

$$\Longrightarrow V^{1\text{-loop}}(\varphi_0) = -\frac{\hbar}{\mathrm{Vol}_{\mathbb{R}^{1,3}}} K^{1\text{-loop}}[\varphi_0] = \frac{\hbar}{2i}\int \frac{d^4k}{(2\pi)^4} \ln\left(\frac{m^2 + \frac{\lambda}{2}\varphi_0^2 - k^2 - i\epsilon}{m^2 - k^2 - i\epsilon}\right)$$

Using the hinted at integral identity backwards, we can rewrite the above,

$$V^{1\text{-loop}}(\varphi_0) = \frac{-\hbar}{2i}\int \frac{d^4k}{(2\pi)^4} \int_{m^2}^{m^2+\frac{\lambda}{2}\varphi_0^2} d\mu^2 \frac{1}{k^2 - \mu^2 - i\epsilon} = \frac{-\hbar}{2i}\int_{m^2}^{m^2+\frac{\lambda}{2}\varphi_0^2} d\mu^2 \int \frac{d^3k}{(2\pi)^3} \int_C \frac{dk^0}{2\pi} \frac{1}{k^0 - \sqrt{\vec{k}^2 + \mu^2 + i\epsilon}}\,\frac{1}{k^0 + \sqrt{\vec{k}^2 + \mu^2 + i\epsilon}},$$

where C is a path along the real axis closed at infinity by a half circle in the upper complex plane, thereby picking up the pole at $k^0 = \sqrt{\vec{k}^2 + X + i\epsilon}$ when applying the residue theorem to perform the k^0-integral. We pick up the contribution,

$$\mathrm{Res}\left(\frac{1}{k^0 - \sqrt{\vec{k}^2 + X + i\epsilon}}, \sqrt{\vec{k}^2 + X + i\epsilon}\right) = \lim_{k^0 \to \sqrt{\vec{k}^2 + X + i\epsilon}} \left(k^0 - \sqrt{\vec{k}^2 + X + i\epsilon}\right)\frac{1}{k^0 - \sqrt{\vec{k}^2 + X + i\epsilon}} = 1,$$

09.05.2015

and no change in sign, because C is positively oriented, so that

$$V^{1\text{-loop}}(\varphi_0) = \frac{-\hbar}{2i}\int\frac{d^3k}{(2\pi)^3}\int_{m^2}^{m^2+\frac{\lambda}{2}\phi_0^2} d\mu^2 \frac{1}{2\pi}\, 2\pi i \frac{1}{\sqrt{\vec{k}^2+\mu^2+i\epsilon}+\sqrt{\vec{k}^2+\mu^2+i\epsilon}} \overset{\epsilon\to 0}{=} -\frac{\hbar}{4}\int\frac{d^3k}{(2\pi)^3}\int_{m^2}^{m^2+\frac{\lambda}{2}\phi_0^2}\frac{d\mu^2}{\sqrt{\vec{k}^2+\mu^2}}$$

$$= -\frac{\hbar}{4}\int\frac{d^3k}{(2\pi)^3}\, 2\sqrt{\vec{k}^2+\mu^2}\,\Big|_{m^2}^{m^2+\frac{\lambda}{2}\phi_0^2} = -\frac{\hbar}{2}\int\frac{d^3k}{(2\pi)^3}\left(\sqrt{\vec{k}^2+m^2+\frac{\lambda}{2}\phi_0^2}-\sqrt{\vec{k}^2+m^2}\right)$$