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Group Theory - Problem Sheet 3

Exercise 1 (Representations of S_n)

a) Use the hook rule to determine the dimension of the irreducible representation associated to the partition

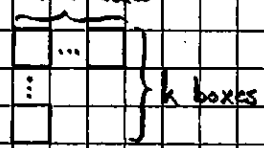
$$\lambda = (n-k, \underbrace{1, \dots, 1}_{k \text{ times}}, 0, \dots, 0)$$

The hook length of a box in a Young tableau is the number of boxes beneath or to the right of the box in the diagram, counting the box itself.

Let ρ^λ be the irreducible repr. of S_n associated with the partition λ, then the hook rule states the dimension of ρ^λ to be $n!$ over the hook lengths in the Young diagram of shape λ, i.e. in this instance

$$\dim(\rho^\lambda) = \frac{n!}{(n-k)+k} = (n-1)!$$

Young diagram:



b) Construct the character table of S_4 by using the Frobenius formula.

According to the Frobenius formula the characters of the irreducible repr. of S_n associated with the partition $\lambda = (\lambda_1, \dots, \lambda_k)$ of n is given by

$$\chi_\lambda(C_i) = \left[\Delta(x) \prod_{j=1} P_j(x)^{i_j}\right]_{(l_1, \dots, l_k)},$$

where P_j is the power sum $P_j(x) = x_1^j + \dots + x_k^j$, k being the number of rows in the Young tableau of λ, and $\Delta(x)$ is the discrimant

$$\Delta(x) = \prod_{1 \le i < j \le k} (x_i - x_j) = \det \begin{pmatrix} 1 & x_k & \cdots & x_k^{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_1 & \cdots & x_1^{k-1} \end{pmatrix}$$

of the set of k independent variables $\{x_1, \dots, x_k\}$. Furthermore, $i = (i_1, \dots, i_n)$ is an n-tuple of non-negative integers s.t. $\sum_r r i_r = n$ so that C_i may denote the conjugacy class consisting of elements with cycle type $(i_1, \dots, i_n)$, and $[f(x)]_{(l_1, \dots, l_k)}$ is the coefficient of $x_1^{l_1} \dots x_k^{l_k}$ in the power series $f(x) = f(x_1, \dots, x_k)$, where $l = (l_1, \dots, l_k)$ is a k-tuple of non-negative integers.

Applying the Frobenius formula yields the following character table

S_4	$C_1 = [(1)]$	$C_2 = [(12)]$	$C_3 = [(123)]$	$C_4 = [(1234)]$	$C_5 = [(12)(34)]$
χ_{triv}	1	1	1	1	1
χ_{sgn}	1	-1	1	-1	1
χ	3	0	0	-1	1
$\chi_{sgn} \otimes \chi$	3	-1	0	1	-1
χ_{other}	2	0	-1	0	2

Exercise 2 (Lie groups and Lie algebras)

f) How are the notions of homomorphisms of Lie groups and Lie algebras related?

Let $\phi: G \to H$ be a Lie group homomorphism. With ϕ_* we denote its derivative at the identity of G. If we pick $\mathfrak{g}, \mathfrak{h}$, the Lie algebras of G and H, as the tangent spaces at their respective identities, then ϕ_* is a map between these two Lie algebras,

$$\phi_* : \mathfrak{g} \to \mathfrak{h},$$

which fulfills all the requirements of a Lie group homomorphism, i.e. it is a linear map that preserves the Lie bracket.

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Exercise 2 (Lie groups and Lie algebras)

a) What is a Lie group?

A Lie group is a group G which is also a differentiable (also called smooth) manifold s.t. the group operations are compatible with the smooth group structure, i.e. s.t.

$$\left.\begin{array}{ll} m: G \times G \longrightarrow G & \text{(multiplication)} \\ (\cdot)^{-1}: G \longrightarrow G & \text{(inversion)} \end{array}\right\} \text{ are smooth maps.}$$

b) What is a Lie algebra?

A Lie algebra is a vector space V with a non-associative, bilinear, skew-symmetric map $[\cdot,\cdot]: V \times V \longrightarrow V$ (called the Lie bracket) which satisfies the Jacobi-identity.

c) What are left-invariant vector fields? Describe them for matrix groups.

A vector field X_v on a Lie group G is left-invariant if

$$(d\mathcal{L}_g)(X_v(h)) = X_v(\mathcal{L}_g(h)) = X_v(gh), \quad \forall g,h \in G$$

where $d\mathcal{L}_g$ is the derivative of the smooth map $\mathcal{L}_g$, and $\mathcal{L}_g$ denotes the group action of the Lie group G defined by left multiplication, i.e. $\mathcal{L}_g: G \longrightarrow G$, $\mathcal{L}_g(h) = gh$ $\forall g,h \in G$ $(\mathcal{L}_e = id_G)$.

d) How does one obtain the structure of a Lie algebra on a tangent space of a Lie group at the identity?

By finding the vector space of all left-invariant vector fields.

Reasoning: Since the vector space of all left-invariant vector fields is isomorphic (as a vector space) to $T_e G$, the tangent of a Lie group G at the identity e, and it in turn is equipped with the structure of a Lie algebra, defined by

$$[v,w] = X_{[v,w]} = [X_v, X_w] \quad \forall v,w \in T_e G,$$

this yields the desired information.

e) Show that the Lie bracket on the tangent space of the identity of matrix groups satisfies the Jacobi identity.

Every matrix (Lie) group is a subgroup of the general linear group $GL(n, \mathbb{K})$

$$GL(n, \mathbb{K}) = \{ A \in Mat(n,n,\mathbb{K}) \mid \det(A) \neq 0 \},$$

i.e. the group of all invertible $n \times n$-matrices over $\mathbb{K}$.

Since the (standard) Lie bracket for Lie algebras of the matrix groups is simply the matrix commutator given by $[A,B] = AB - BA$ for $A, B \in GL(n, \mathbb{K})$, we calculate

$$[A,[B,C]] + [B,[C,A]] + [C,[A,B]] = ABC - BCA - ACB + CBA$$

$$+ BCA - CAB - BAC + ACB + CAB - ABC - CBA + BAC = 0,$$

which holds for all $A, B, C \in GL(n, \mathbb{K})$ and therefore for all possible elements of matrix groups.

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Exercise 3 (SU(2))

Show that $SU(2) = \{A \in Mat(2,2,\mathbb{C}) \mid A^\dagger A = \mathbb{1}_2,\ \det A = 1\}$ can be parametrized in the following way

$$SU(2) = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \,\middle|\, a,b \in \mathbb{C},\ |a|^2 + |b|^2 = 1 \right\}.$$

(Hence, SU(2) is homeomorphic to the three sphere S^3, and in particular simply connected.)

We write an element $A \in SU(2)$ as $A = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$ with $a,b,c,d \in \mathbb{C}$. Then

$$A^\dagger A = \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} |a|^2 + |b|^2 & \bar{a}c + \bar{b}d \\ a\bar{c} + b\bar{d} & |c|^2 + |d|^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The above property of A as an element $SU(2)$ yields the following three constraints on $a,b,c,d \in \mathbb{C}$,

i) $|a|^2 + |b|^2 = 1$ ii) $|c|^2 + |d|^2 = 1$ iii) $a\bar{c} + b\bar{d} = 0$

The condition $\det(A) = 1$ further supplies

iv) $ad - bc = 1$

From iii), $a = -\frac{b\bar{d}}{\bar{c}}$ follows. Inserting into i), we obtain

$$\frac{|b|^2|d|^2}{|c|^2} + |b|^2 = 1 \implies \frac{|c|^2}{|b|^2} = |c|^2 + |d|^2 = 1,$$

i.e. $|b| = |c|$. Similarly, we find $|a| = |d|$ by subtracting i) and ii)

$$|a|^2 + \underbrace{|b|^2 - |c|^2}_{0} - |d|^2 = |a|^2 - |d|^2 = 0.$$

Remark: We assumed above, that $b \neq 0$. If $b = 0$, it becomes very simple. By iv), we know $ad = 1$ and by iii), $a\bar{c} = 0$. The latter implies $c = 0$, since $a \neq 0$, and A becomes diagonal with $a = \bar{d}$ because $ad = 1$.

Now, returning to the case $b \neq 0$, we discover that our findings are captured by the ansatz

$$a = e^{i\alpha}\cos\theta, \quad b = e^{i\beta}\sin\theta, \quad c = -e^{i\gamma}\sin\theta, \quad d = e^{i\delta}\cos\theta,$$

where $\alpha, \beta, \gamma, \delta, \theta \in \mathbb{R}$. Applying iv) to this ansatz,

$$ad - bc = e^{i(\alpha+\delta)}\cos^2\theta + e^{i(\beta+\gamma)}\sin^2\theta = 1 \iff \alpha = -\delta, \ \beta = -\gamma,$$

we finally arrive at

$$a = e^{i\alpha}\cos\theta = e^{-i\delta}\cos\theta = \bar{d}, \quad b = e^{i\beta}\sin\theta = e^{-i\gamma}\sin\theta = -\bar{c}.$$

Thus, $SU(2)$ may be parametrized by

$$SU(2) = \left\{ \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \,\middle|\, a, b \in \mathbb{C}, \ |a|^2 + |b|^2 = 1 \right\}$$

Exercise 4 ($SU(2)$ versus $SO(3)$)

a) Determine the Lie algebras of $SU(2)$ and $SO(3)$ and show that they are isomorphic.

The $SU(2)$ Lie algebra is

$$su(2) = \{ A \in Mat(2,2,\mathbb{C}) \mid tr(A) = 0, \ A^\dagger = -A \}$$
$$= \left\{ \begin{pmatrix} ia & -\bar{z} \\ z & -ia \end{pmatrix} \,\middle|\, a \in \mathbb{R}, \ z \in \mathbb{C} \right\},$$

which amounts to the vector space of all traceless, antihermitian 2×2-matrices over $\mathbb{C}$. It is $n^2 - 1 = 3$-dimensional and generated by the following matrices

$$u_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}.$$

The Lie algebra of $SO(3)$,

$$so(3) = \{ A \in Mat(3,3,\mathbb{R}) \mid A^T = -A \} = \left\{ \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \middle| \omega_x, \omega_y, \omega_z \in \mathbb{R} \right\}$$

is the vector space of all antisymmetric 3×3-matrices over $\mathbb{R}$.

$so(3)$ is $\frac{n}{2}(n-1) = 3$-dimensional with a generating basis formed by the following three matrices

$$o_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \quad o_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad o_3 = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Both assertions concerning the structure of the Lie algebras $su(2)$ and $so(3)$ can be seen by selecting and differentiating smooth curves from their corresponding Lie groups:

- Let $U(t) \in SU(2)$ be a smooth curve with $U(0) = \mathbb{1}_2$. Then $U(t)$ satisfies

$$U^{\dagger}(t)\, U(t) = \mathbb{1}_2, \qquad \det(U(t)) = 1.$$

Differentiation yields

$$\frac{dU^{\dagger}(t)}{dt} U(t) + U^{\dagger}(t) \frac{dU(t)}{dt} = 0, \qquad tr\left(U^{-1}(t) \frac{dU(t)}{dt}\right) = 0,$$

which, for $t=0$, and writing $\left.\frac{dU(t)}{dt}\right|_{t=0}$ as $u \in su(2)$, demonstrates the above claims

$$u^{\dagger} + u = 0, \qquad tr(u) = 0.$$

- Let $O(t) \in SO(3)$ be a smooth curve with $O(0) = \mathbb{1}_3$. Then $O(t)$ satisfies the defining relation of $O(3)$-elements $O^T(t)\, O(t) = O$. Thus

$$\frac{dO^T(t)}{dt} O(t) + O^T(t) \frac{dO(t)}{dt} = 0$$

and so for $t=0$ and $o := \left.\frac{dO(t)}{dt}\right|_{t=0}$, $o^T + o = 0$.

Now on to the isomorphy between su(2) and so(3). We can show this relation using the group homomorphism $\Phi : SU(2) \to SO(3)$ to construct an isomorphism $\phi : su(2) \longrightarrow su(3)$.

We start by expressing Φ using abstract index notation. Let $f(\vec{x})$ be a function of position $\vec{x}$, $\underline{R} \in SO(3)$ a rotation matrix and $\underline{U}$, a unitary transformation, the image of $\Phi(\underline{R})$ in $SU(2)$. Then

$$f(\underline{R}\,\vec{x}) = f(\Phi(\underline{U})\,\vec{x}) = \underline{U}\, f(\vec{x})\, \underline{U}^\dagger,$$

$$\underline{R}^i{}_j\, x^j\, \underline{\sigma}_i = \underline{U}\, x^i\, \underline{\sigma}_i\, \underline{U}^\dagger.$$

Differentiating at $t=0$ and setting $\left.\frac{d\underline{R}}{dt}\right|_{t=0} = \underline{X} \in so(3)$, $\underline{U}(0) = \mathbb{1}_2$, and $\left.\frac{d\underline{U}}{dt}\right|_{t=0} = \underline{A} = a^i \underline{s}_i \in su(2)$, we obtain

$$\underline{X}^i{}_j\, x^j\, \underline{\sigma}_i = \underline{A}\, x^j\, \underline{\sigma}_j + x^j\, \underline{\sigma}_j\, \underline{A}^\dagger$$

$$x^j\, \underline{X}^i{}_j\, \underline{\sigma}_i = x^j(\underline{A}\,\underline{\sigma}_j - \underline{\sigma}_j\,\underline{A}) = x^j[\underline{A}, \underline{\sigma}_j] = x^j[a^i \underline{s}_i, \underline{\sigma}_j]$$

$$= -\frac{i}{2} x^j a^i [\underline{\sigma}_i, \underline{\sigma}_j] = -\frac{i}{2} x^j a^i (2i\,\epsilon_{ij}{}^k\, \underline{\sigma}_k)$$

$$= x^j a^i\, \epsilon_{ij}{}^k\, \underline{\sigma}_k = x^j(-a^k\, \epsilon_{kj}{}^i)\, \underline{\sigma}_i$$

$$=: x^j\, \phi(\underline{A})_j{}^i\, \underline{\sigma}_i$$

So the isomorphism between su(2) and so(3) is given by

$$\phi(\underline{A})^i{}_j = \epsilon^i{}_{jk}\, a^k : su(2) \longrightarrow so(3).$$

With this knowledge, we can say that because Lie algebras are locally identical to their corresponding Lie groups, SU(2) and SO(3) are locally isomorphic.