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Group Theory - Problem Sheet 2

Exercise 1 (Representations)

Recall the following notions from the lecture:

a) What is a representation?

A (linear) representation of a group G on a vector space V over a field K is a group homomorphism from G to $GL(V)$, i.e. a map

$$\rho: V \longrightarrow GL(V) = Aut(V) = \{\text{all invertible, linear transf. } V \longrightarrow V\}$$

such that $\rho(g_1)\rho(g_2) = \rho(g_1 g_2) \quad \forall g_1, g_2 \in G$.

V is called the representation space and $\dim(V)$ the dimension of the representation.

b) What is a homomorphism (intertwiner) between two repr.?

A group homomorphism between two groups is a map that preserves the group operation, i.e. given two groups $(G, \cdot)$ and $(H, *)$, a group homomorphism φ with

$$\varphi: G \rightarrow H$$

fulfills $\varphi(g_1 \cdot g_2) = \varphi(g_1) * \varphi(g_2) \quad \forall g_1, g_2 \in G$.

Let V_1 and V_2 be the representation spaces of two (linear) representations of the group G,

$$\rho_1 : G \rightarrow GL(V_1), \quad \rho_2 : G \rightarrow GL(V_2).$$

Then, a linear map $f: V_1 \rightarrow V_2$ is called an intertwiner of the representations ρ_1 and ρ_2 if it commutes under the group action of G, i.e. if $f \circ \rho_1(g) = \rho_2(g) \circ f \quad \forall g \in G$.

An intertwiner's property can be visualized by the following diagram.

$$\begin{tikzcd} V_1 \arrow[r, "\rho_1(g)"] \arrow[d, "f"'] & V_1 \arrow[d, "f"] \\ V_2 \arrow[r, "\rho_2(g)"] & V_2 \end{tikzcd}$$

Notably, an intertwiner must fulfill this diagram for every $g \in G$.
If f is invertible (only possible if $\dim(V_1) = \dim(V_2)$), we have $\rho_1(g) = f^{-1} \circ \rho_2(g) \circ f$ and the representations ρ_1 and ρ_2 are said to be equivalent.

c) What is a subrepresentation?

A subspace $W \subset V$ of the representation space V of a representation ρ is called a subrepresentation, if it is invariant under the group action of the represented group G, where invariant means that $gw \in W \; \forall g \in G$ and $\forall w \in W$.
The restriction of ρ to W, $\rho(W)|_W$, is also refered to as subrepresentation, if $W \subset V$ is invariant subspace. $\rho|_W : G \to GL(W)$ is then a repr. on its own.

d) What are reducible and irreducible representations?

A representation $\rho: G \to GL(V)$ is said to be irreducible if it has only the trivial subrepresentations $\{0\}$ and V. If there is non-trivial subrepresentation, ρ is reducible.

e) Recall Schur's lemma.

If ρ_1 and ρ_2 are two finite-dimensional irreducible representations of a group G and $f: V_1 \to V_2$ is an intertwiner between their representation spaces, then either $f \equiv 0$ or f is invertible i.e. an isomorphism.
In the special case of $V_1 \cong V_2$, it further yields $f = \lambda \, id_V$ (with $\lambda \in \mathbb{C}$, if V_1 and V_2 are vector spaces over $\mathbb{C}$).

02.06.2015

f) Argue that complex representations of finite groups are fully decomposable.

A representation $\rho: G \to GL(V)$ of a group G is called fully-decomposable, if it can be decomposed into a direct sum of irreducible representations, $\rho = \rho_1 \oplus \dots \oplus \rho_n$ where ρ_i, $i \in \{1, \dots, n\}$ each irreducible.

The above claim follows from this argument:

Let $\rho: G \to U(V)$ be a reducible complex (also called unitary) representation (if it were irreducible, it would be fully-decomposable by definition).

$\Rightarrow \exists\, W \subset V$ invariant subspace

We can define $W^\perp = \{v \in V \mid \langle v, w \rangle = 0 \;\forall w \in W\}$.

Then, since

$$\forall g \in G,\ \forall w \in W,\ \forall v \in W^\perp: \quad 0 = \langle v, w \rangle \overset{\substack{W \text{ invariant subspace} \\ \downarrow}}{=} \langle v, \rho(g)\, w \rangle \overset{\substack{\rho \text{ unitary repr.} \\ \downarrow}}{=} \langle \rho(g^{-1}) v, w \rangle,$$

we know that $W^\perp$ is an invariant subspace as well.

We infer that $\rho = \rho|_W \oplus \rho|_{W^\perp}$

still unitary, therefore we may split again.

The above scheme can be applied repeatedly until the subrepresentations become irreducible. Finally

$$\rho = \bigoplus_i \rho_i \quad , \quad \text{with } \rho_i \text{ irreducible } \forall i$$

One may say that irreducible repr. are the building blocks of unitary repr.

Exercise 2 (Representations of abelian groups)

a) Show that all irreducible representations of a finite Group G are one-dimensional if and only if G is abelian. (Hint: What is the sum of the squares of the dimensions of the irreducible repr.?)

To be shown: $\dim(\rho_i) = 1 \quad \forall \rho_i : G \longrightarrow GL(V_i)$ irreducible, $i \in \{1, \dots, n\}$

$$\Longleftrightarrow \quad G \text{ abelian}$$

"$\Longrightarrow$": Assume all irreducible repr. have $d_i = \dim(\rho_i) = 1$. Then we have

$$|G| \overset{!}{=} \sum_{i=1}^{n} d_i^2 = \sum_{i=1}^{n} 1 = n, \quad \text{with } n\text{: number of irred. repr.}$$

We also have that

$$\sum_{i=1}^{n} d_i \overset{!}{=} \text{'number of conjugacy classes of } G\text{'} = n$$

Therefore, 'number of conjugacy classes' $= n = |G|$, i.e. every element of G forms its own conjugacy class.

$$\Longleftrightarrow \quad hgh^{-1} = g \quad \forall g, h \in G \quad \Longrightarrow \quad hg = gh \quad \forall g, h \in G \quad \Longleftrightarrow \quad G \text{ is abelian}$$

"$\Longleftarrow$": Assume G is abelian $\Longleftrightarrow gh = hg \quad \forall g, h \in G$. Then, since every repr. is a homomorphism, we have

$$\rho(g_1)\rho(g_2) \overset{\rho \text{ homomorphism}}{=} \rho(g_1 g_2) \overset{G \text{ abelian}}{=} \rho(g_2 g_1) = \rho(g_2)\rho(g_1)$$

Applying the special case of Schur's lemma, which states that if $f: V \longrightarrow V$ commutes with $\rho(g)$ $\forall g \in G$, i.e.

$$f \circ \rho(g) = \rho(g) \circ f \quad \forall g \in G \quad \Longrightarrow \quad f = \lambda \, id_V \quad \text{for some } \lambda \in \mathbb{C},$$

we obtain $\rho(g) = \lambda_g \, id_V = \begin{pmatrix} \lambda_g & & 0 \\ & \ddots & \\ 0 & & \lambda_g \end{pmatrix} = \bigoplus_i \rho_i$, with ρ_i irreducible repr. of $\dim(\rho_i) = 1$.

b) What are the irreducible representations of the cyclic group $\mathbb{Z}_n$?

Since $\mathbb{Z}_n$ is abelian, we know from part a) that every irreducible subrepr. of $\mathbb{Z}_n$ is one-dimensional and that there are exactly $|\mathbb{Z}_n| = n$ of them.

Let $id_{\mathbb{C}}^{1/n}$ denote the n-th root of unity in $\mathbb{C}$, then the map

$$\rho : \mathbb{Z}_n \longrightarrow \mathbb{C}^*, \quad \rho(g^j) = id_{\mathbb{C}}^{j/n} \quad \forall g^j \in \mathbb{Z}_n$$

is a group homomorphism. Since $id_{\mathbb{C}}^{\frac{1}{k}}$, $k \in \{1, \dots, n\}$ are the first through n-th roots of unity in $\mathbb{C}$, the n irreducible subrepresentations of $\mathbb{Z}_n$ with dimension one are

$$\rho_k : G \longrightarrow \mathbb{C}^*, \quad \rho_k(g^j) = id_{\mathbb{C}}^{j/k} = e^{2\pi i \frac{j}{k}}, \quad \text{with } k \in \{1, \dots, n\}.$$

Exercise 3 (Orthogonality relation)

Let (ρ_1, V_1) and (ρ_2, V_2) be two irreducible (finite-dimensional complex) representations of a finite group G. Choose bases of V_1 and V_2 and let $(\rho_1(g))_{ij}$ and $(\rho_2(g))_{ab}$ be the repr. matrices with respect to these bases. Then use Schur's lemma to show:

a) If ρ_1 and ρ_2 are inequivalent, then

$$\sum_{g \in G} (\rho_1(g^{-1}))_{ij} \, (\rho_2(g))_{ab} = 0$$

for all i, j, a, b.

To show this, we consider the matrix

$$M_X = \sum_{g \in G} \underbrace{(\rho_1(g))^{-1}}_{\rho_1(g^{-1})} X \rho_2(g),$$

where X is an arbitrary matrix with d_1 rows and d_2 columns, d_1 and d_2 being the dimensionalities of the representations

ρ_1 and ρ_2, s.t. M_X is a $d_1 \times d_1$ - matrix.

Now multiplying M_X from the left and right with arbitrary matrices in the representations ρ_1 and ρ_2 respectively,

$$\rho_1^{-1}(h)\, M_X\, \rho_2(h) = \sum_{g\in G} \underbrace{\rho_1^{-1}(h)}_{\rho_1(h^{-1})}\, \underbrace{\rho_1^{-1}(g)}_{\rho_1(g^{-1})}\, X\, \rho_2(g)\, \rho_2(h) \qquad \text{for some } h\in G$$

$$= \sum_{g\in G} \rho_1(\underbrace{h^{-1}g^{-1}}_{(gh)^{-1}})\, X\, \rho_2(gh) \qquad g \longmapsto gh^{-1}$$

$$= \sum_{g\in G} \rho_1(g^{-1})\, X\, \rho_2(g) = M_X,$$

we find that M_X is an intertwiner between the repr. ρ_1 and ρ_2, i.e. $\rho_1(g)\, M_X = M_X\, \rho_2(g)$.

Since ρ_1 and ρ_2 are taken to be inequivalent, i.e.

$$\nexists\, M_X : \quad \rho_1(g) = M_X\, \rho_2(g)\, M_X^{-1}, \qquad (M_X \text{ not invertible})$$

we know by Schur's lemma that $M_X \equiv 0$.

Therefore, writing element-wise, we have

$$0 = (M_X)_{ib} = \sum_{g\in G} (\rho_1(g^{-1}))_{ik}\, X_{kl}\, (\rho_2(g))_{lb}.$$

Since the matrix X was arbitrary, we may choose any form.

We pick one X for every tupel of j, a given by

$$(X_{kl})_{ja} = \delta_{kj}\, \delta_{la}$$

to conclude that

$$\sum_{g\in G} (\rho_1(g^{-1}))_{ik}\, \delta_{kj}\, \delta_{la}\, (\rho_2(g))_{lb} = \sum_{g\in G} (\rho_1(g^{-1}))_{ij}\, (\rho_2(g))_{ab} \stackrel{!}{=} 0$$

for all i, j, a, b.

02.06.2015

b) If on the other hand $\rho_1 = \rho_2$, then

$$\frac{1}{|G|}\sum_{g\in G} (\rho(g^{-1}))_{ij} (\rho_2(g))_{ab} = \frac{1}{\dim(V_1)} \delta_{ib}\delta_{ja}.$$

In the case of $\rho_1 = \rho_2 =: \rho$, ρ is still an irreducible representation of G and M_X with

$$M_X = \sum_{g\in G} (\rho(g))^{-1} X \rho(g)$$

is still an intertwiner, so that we may apply Schur's lemma for the special case of $V_1 = V_2 =: V$ to conclude

$$M_X = \lambda \, id_V \quad \text{for some } \lambda \in \mathbb{C} \text{ for every } g\in G$$

Taking the trace of M_X, we obtain

$$Tr(M_X) = Tr_V\Big(\sum_{g\in G} \rho(g^{-1}) X \rho(g)\Big) \overset{\text{linearity of trace}}{=} \sum_{g\in G} Tr_V\big(\rho(g^{-1}) X \rho(g)\big)$$

$$\overset{\text{cyclicity of trace}}{=} \sum_{g\in G} Tr_V\big(\underbrace{\rho(g)\rho(g^{-1})}_{\rho(gg^{-1}) = \rho(e) = id_V} X\big) = \sum_{g\in G} Tr_V(\underbrace{id_V X}_{X}) = |G|\, Tr_V(X)$$

$$Tr(M_X) = Tr(\lambda\, id_V) = \lambda\, Tr(id_V) = \lambda \dim(V) \implies \lambda = \frac{|G|}{\dim(V)} Tr_V(X)$$

Choosing again $(X_{ja})_{kl} = \delta_{kj}\delta_{al}$, we obtain

$$(M_X)_{ib} = \sum_{g\in G} (\rho(g^{-1}))_{ik}\, \delta_{kj}\, \delta_{al}\, (\rho(g))_{lb} = \sum_{g\in G} (\rho(g^{-1}))_{ij}\, (\rho(g))_{ab}$$

$$= \lambda \delta_{ib} = \frac{|G|}{\dim(V)} Tr_V(X)\, \delta_{ib} = \frac{|G|}{\dim(V)} \underbrace{Tr_V(\delta_{kj}\delta_{al})}_{\delta_{kj}\delta_{ak} = \delta_{ja}} = \frac{|G|}{\dim(V)} \delta_{ib}\delta_{ja}$$

$$\implies \frac{1}{|G|}\sum_{g\in G} (\rho(g^{-1}))_{ij} (\rho(g))_{ab} = \frac{1}{\dim(V)} \delta_{ib}\delta_{ja}$$

c) Deduce that

$$\frac{1}{|G|}\sum_{g\in G}\chi_{\rho_1}(g^{-1}h)\,\chi_{\rho_2}(g)=\begin{cases}\frac{\chi_{\rho_1}(h)}{\dim(\rho_1)}, & \text{if } \rho_1 \hat{=} \rho_2 \\ 0, & \text{otherwise}\end{cases}.$$

We express the characters $\chi_{\rho_1}(g^{-1}h) := Tr_{V_1}(\rho_1(g^{-1}h))$ and $\chi_{\rho_2}(g) := Tr_{V_2}(\rho_2(g))$ as

$$\chi_{\rho_1}(g^{-1}h)=\sum_{i,j}(\rho_1(g^{-1}))_{ij}\,(\rho_1(h))_{ji} \quad \text{and}$$

$$\chi_{\rho_2}(g)=\sum_k (\rho_2(g))_{kk}$$

to obtain

$$\frac{1}{|G|}\sum_{g\in G}\chi_{\rho_1}(g^{-1}h)\,\chi_{\rho_2}(g)=\frac{1}{|G|}\sum_{g\in G}\sum_{i,j,k}(\rho_1(g^{-1}))_{ij}\,(\rho_1(h))_{ji}\,(\rho_2(g))_{kk}$$

$$=\frac{1}{|G|}\sum_{i,j,k}\sum_{g\in G}(\rho_1(g^{-1}))_{ij}\,(\rho_2(g))_{kk}\,(\rho_1(h))_{ji}$$

$$=\begin{cases}\frac{1}{\dim(V)}\sum_{i,j,k}\delta_{ik}\,\delta_{jk}\,(\rho(h))_{ji}=\frac{1}{\dim(V)}\sum_k(\rho(h))_{kk}=\frac{\chi_\rho(h)}{\dim(V)} & \text{if } \rho_1=\rho_2=:\rho \text{ using a)} \\ \frac{1}{|G|}\sum_{i,j,k}0\cdot(\rho(h))_{ji}=0 & \text{otherwise using a)}\end{cases}$$

<u>Exercise 4 (Symmetric and antisymmetric products)</u>

Let (ρ, V) be a finite dimensional representation. Consider the tensor product representation $\rho\otimes\rho$ on $V\otimes V$. Define $\sigma: V\otimes V \longrightarrow V\otimes V$ by $\sigma(v_1\otimes v_2)=v_2\otimes v_1 \;\forall v_1, v_2\in V$.

a) Show that the tensor product decomposes into the sum $V\otimes V = S^2V \oplus \Lambda^2 V$ of spaces of symmetric and anti-symmetric tensors $S^2V=\{v\in V \mid \sigma(v)=v\}$ and $\Lambda^2V=\{v\in V\mid \sigma(v)=-v\}$, and that these are invariant subspaces in $V\otimes V$. (Hint: Consider the projectors $\frac{1}{2}(1\pm\sigma): V\otimes V\longrightarrow V\otimes V$.)

Following the hint, we examine $\pi_\pm := \frac{1}{2}(id_{V\otimes V} \pm \sigma) : V\otimes V \to V\otimes V$

$$\pi_\pm^2 = \frac{1}{4}(id_{V\otimes V} \pm \sigma)^2 = \frac{1}{4}(\underbrace{id_{V\otimes V}^2}_{id_{V\otimes V}} \pm 2\underbrace{id_{V\otimes V}\,\sigma}_{\sigma} + \underbrace{\sigma^2}_{id_{V\otimes V}}) = \frac{1}{4}(2\,id_{V\otimes V} \pm 2\sigma) = \pi_\pm$$

As hinted, we find $\pi_\pm$ to be projectors. In addition,

$$\pi_+ + \pi_- = \frac{1}{2}(id_{V\otimes V} + \sigma) + \frac{1}{2}(id_{V\otimes V} - \sigma) = id_{V\otimes V}.$$

Therefore, the tensor product $V\otimes V$ can be decomposed as

$$V\otimes V = \operatorname{Im}(\pi_+) \oplus \operatorname{Im}(\pi_-)$$

It remains to be shown that $\operatorname{Im}(\pi_+) = S^2V$ and $\operatorname{Im}(\pi_-) = \Lambda^2 V$.

$S^2V \subset \operatorname{Im}(\pi_+)$: $\forall v \in S^2V : \sigma(v) = v$

$$\Longrightarrow \forall v \in S^2V : \pi_+(v) = \frac{1}{2}(id_{V\otimes V}(v) + \sigma(v)) = \frac{1}{2}(v+v) = v$$

$$\Longrightarrow S^2V \subset \operatorname{Im}(\pi_+), \text{ since } \pi_+ \text{ is projector}$$

$\operatorname{Im}(\pi_+) \subset S^2V$: $\forall v \in \operatorname{Im}(\pi_+) : \pi_+(v) = v$, since π_+ is projector

$$\Longrightarrow \pi_+(v) = \frac{1}{2}(id_{V\otimes V}(v) + \sigma(v)) = \frac{1}{2}(v + \sigma(v)) \stackrel{!}{=} v \iff \sigma(v) = v$$

$$\Longrightarrow \operatorname{Im}(\pi_+) \subset S^2V$$

$$\Longrightarrow \operatorname{Im}(\pi_+) = S^2V$$

This goes through analogously for $\operatorname{Im}(\pi_-) = \Lambda^2 V$.

$$\Longrightarrow V\otimes V = S^2V \oplus \Lambda^2 V$$

To further show that S^2V and Λ^2V are invariant subspaces in $V\otimes V$, we remember a remark from the lecture, which states that image and kernel of every intertwiner form invariant subspaces. Since σ is an intertwiner because it fulfills $\rho\otimes\rho(g)\circ\sigma = \sigma\circ\rho\otimes\rho(g)$, so is $\pi_\pm$ w.r.t. $\rho\otimes\rho$. Therefore, S^2V and Λ^2V are invariant subspaces.

b) Show that the characters of the corresponding subrepresentations can be expressed as $\chi_{S^2V}(g) = \frac{1}{2}[(\chi_V(g))^2 + \chi_V(g^2)]$, $\chi_{\Lambda^2V}(g) = \frac{1}{2}[(\chi_V(g))^2 - \chi_V(g^2)]$. (A character is defined as the trace over a representation $\chi_V(g) = tr_V(\rho(g))$.)

A subrepresentation is formed by restricting a represention ρ to a subspace W of its representation space V, $\rho|_W$. Since $\pi_\pm$ are projectors onto their respective subspaces, we may write the subrepresentations $\rho\otimes\rho|_{S^2V}$ and $\rho\otimes\rho|_{\Lambda^2V}$ as

$$\rho\otimes\rho|_{S^2V} = \rho\otimes\rho\,\pi_+ , \qquad \rho\otimes\rho|_{\Lambda^2V} = \rho\otimes\rho\,\pi_-$$

Making use of $\pi_\pm$ also being intertwiners, we may express the characters by

$$\chi_{S^2V}(g) = tr_{V\otimes V}(\pi_+\, \rho\otimes\rho(g)) = \frac{1}{2}\Big[tr_{V\otimes V}(id_{V\otimes V}\, \rho\otimes\rho(g)) + tr_{V\otimes V}(\sigma\, \rho\otimes\rho(g))$$

$$\chi_{\Lambda^2V}(g) = tr_{V\otimes V}(\pi_-\, \rho\otimes\rho(g)) = \frac{1}{2}\Big[tr_{V\otimes V}(id_{V\otimes V}\, \rho\otimes\rho(g)) - tr_{V\otimes V}(\sigma\, \rho\otimes\rho(g))\Big]$$

We can simplify both summands

$$tr_{V\otimes V}(id_{V\otimes V}\, \rho\otimes\rho(g)) = tr_V(\rho(g))\, tr_V(\rho(g)) = [tr_V(\rho(g))]^2 = \chi_V^2(g)$$

To calculate the second, we need to choose a basis, say e_i, for V. Then

$$tr_{V\otimes V}(\sigma\, \rho\otimes\rho(g)) = tr_{V\otimes V}(\sigma\, \rho(g)\otimes\rho(g)\, (e_i\otimes e_j))$$

$$= tr_{V\otimes V}(\rho(g)\otimes\rho(g)\,(e_j\otimes e_i)) = tr_{V\otimes V}(\rho_{ai}(g)\,\rho_{bj}(g)\, e^b\otimes e^a)$$

$$= tr_V(\rho(g)\,\rho(g)) = tr_V(\rho(g^2))$$

02.06.2015

Reinserting the simplifications into $\chi_{S^2V}(g)$ and $\chi_{\Lambda^2V}(g)$, we arrive at the desired result:

$$\chi_{S^2V}(g) = \frac{1}{2}\left[\chi_V^2(g) + \chi_V(g^2)\right]$$

$$\chi_{\Lambda^2V}(g) = \frac{1}{2}\left[\chi_V^2(g) - \chi_V(g^2)\right]$$

Exercise 5 (Representations of D_5)

Consider the dihedral group

$$D_5 = \{e = R_0, R_1, R_2, R_3, R_4, S_0, S_1, S_2, S_3, S_4\}$$

of symmetries of the regular pentagon in the plane. Its character table is given by

D_5	$C_0 = \{e\}$	$C_1 = \{R_1, R_4\}$	$C_2 = \{R_2, R_3\}$	$C_3 = \{S_0, \dots, S_5\}$
χ_0	1	1	1	1
χ_1	1	1	1	-1
χ_2	2	α	β	0
χ_3	2	β	α	0

a) Use orthogonality of characters to show that $\alpha = \frac{1}{2}(-1+\sqrt{5})$ and $\beta = \frac{1}{2}(-1-\sqrt{5})$.

The orthogonality relation of two characters is given by

$$\sum_a \overline{\chi_i(C_a)}\,\chi_j(C_a)\frac{|C_a|}{|G|} = \delta_{ij}, \quad \text{where} \quad \overline{\chi(g)} = \chi(g^{-1})$$

However, using in this example the orthogonality of rows would yield the equally correct but undesired solution $\alpha = -3 \pm \sqrt{3}$, $\beta = -1 \pm \sqrt{3}$.

So we apply the orthog. rel. for columns given by

$$\sum_i \overline{\chi_i(C_a)}\,\chi_i(C_b) = \frac{|G|}{|C_a|}\delta_{ab}$$

Multiplying the first and second (or third), and the second and third columns, we obtain the constraints

$$1\cdot 1 + 1\cdot 1 + 2\alpha + 2\beta = 2 + 2\alpha + 2\beta = 0 \iff 1+\alpha+\beta = 0,$$

$$1\cdot 1 + 1\cdot 1 + \alpha\beta + \beta\alpha = 2 + 2\alpha\beta = 0 \iff 1+\alpha\beta = 0$$

$$\implies \beta = -(1+\alpha), \quad 1+\alpha\beta = 1-\alpha(1+\alpha) = 1-\alpha-\alpha^2 = 0$$

$$\iff \left(\alpha+\frac{1}{2}\right)^2 = \frac{5}{4} \implies \alpha = -\frac{1}{2} \pm \sqrt{\frac{5}{4}} = \frac{1}{2}\left(-1 \pm \sqrt{5}\right)$$

$$\implies \beta = -\left(1 + \frac{1}{2}\left(-1 \pm \sqrt{5}\right)\right) = \frac{1}{2}\left(-1 \mp \sqrt{5}\right)$$

b) Consider the following representations $\rho_2 \otimes \rho_3$, $\Lambda^2 \rho_2$, $S^2 \rho_2$. Determine their characters and decompose them into irreducible representations.

Characters are multiplicative with respect to tensor product, i.e.

$$\chi_{\rho_2 \otimes \rho_3}(g) = \chi_{\rho_2}(g) \cdot \chi_{\rho_3}(g)$$

Applying our result from exercise 4.b), i.e.

$$\chi_{S^2 V}(g) = \frac{1}{2}\left(\chi_V^2(g) + \chi_V(g^2)\right), \quad \chi_{\Lambda^2 V}(g) = \frac{1}{2}\left(\chi_V^2(g) - \chi_V(g^2)\right),$$

we may write a new character table for $\chi_{\rho_2\otimes\rho_3}$, $\chi_{S^2\rho_2}$, $\chi_{\Lambda^2\rho_2}$

D_5	C_0	C_1	C_2	C_3
$\chi_{\rho_2\otimes\rho_3}$	4	-1	-1	0
$\chi_{S^2\rho_2}$	3	$-\alpha$	$-\beta$	1
$\chi_{\Lambda^2\rho_2}$	1	1	1	-1

Apparently, $\chi_{\rho_2\otimes\rho_3} = \chi_2 + \chi_3$. Therefore, $\rho_2 \otimes \rho_3 \cong \rho_2 \oplus \rho_3$. Also

$$\chi_{S^2\rho_2} = \chi_0 + \chi_3 \implies S^2\rho_2 \cong \rho_0 \oplus \rho_3$$

$$\chi_{\Lambda^2\rho_2} = \chi_1 \implies \Lambda^2\rho_2 \cong \rho_1$$