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Group Theory – Problem Sheet 1

Exercise 1 (Definitions)

a) Define what a group is and show from the axioms that left-identity and -inverses are also right-identity and -inverses, and that identity and inverses are unique.

group: algebraic structure $(G, \langle\cdot,\cdot\rangle)$ consisting of

- a set of elements G and
- an operation $\langle\cdot,\cdot\rangle$ that satisfies the

group axioms

1. Closure: $\forall a, b \in G \quad \langle a, b\rangle \in G$
2. Associativity $\forall a, b, c \in G \quad \langle\langle a, b\rangle, c\rangle = \langle a, \langle b, c\rangle\rangle$
3. Identity $\quad \exists! e \in G \; \forall a \in G: \; \langle e, a\rangle = a$
4. Inverse $\quad \exists! a^{-1} \in G \; \forall a \in G: \; \langle a^{-1}, a\rangle = e$

From now on, the group operation between two elements of a group is implicit, i.e. $\langle a, b\rangle \longrightarrow ab$. We prove that left- and right-identity and inverse are equal and their uniqueness:

$$a^{-1}a = e \quad \Longrightarrow \quad a^{-1}(a a^{-1}) = (\underbrace{a^{-1}a}_{e})a^{-1} = e a^{-1} = a^{-1} \quad \Longrightarrow \quad a a^{-1} = e \quad \text{(right inverse)}$$

$$a a^{-1} = e \quad \Longrightarrow \quad ae = a(a^{-1}a) = (\underbrace{a a^{-1}}_{e})a = ea \quad \Longrightarrow \quad ea = a \overset{!}{=} ea \quad \text{(right identity)}$$

$$\text{assume } \exists e, e' \in G: \; ea = a, \; e'a = a \;\; \forall a \in G \Longrightarrow e = ee' = e' \quad \Longrightarrow \quad e = e' \quad \text{(identity uni.)}$$

$$\text{assume } \exists a^{-1}, a'^{-1} \in G: \; a^{-1}a = e, \; a'^{-1}a = e \;\; \forall a \in G \Longrightarrow a^{-1}a = e = a'^{-1}a \Longrightarrow a^{-1} = a'^{-1} \quad \text{(inverse uni.)}$$

b) What is a group homomorphism? Show that the map $\mathbb{Z} \to \mathbb{Z}_n$, $z \mapsto [z]$ is a group homomorphism.

group homomorphism: given two groups $(G,*)$, $(H,\cdot)$, $\varphi: G \to H$ is a group homomorphism if $\varphi(g_1 * g_2) = \varphi(g_1) \cdot \varphi(g_2)$ $\forall g_1, g_2 \in G$

$\forall z_1, z_2 \in \mathbb{Z}$: $[z_1 + z_2] = [z_1] + [z_2]$

c) What is a subgroup? Show that the rotations form a subgroup $\mathbb{Z}_n \subset D_n$ of the dihedral group D_n.

given two groups $(G,\cdot)$, $(H,\cdot)$, $(G,\cdot)$ is a subgroup of $(H,\cdot)$ if $\forall g \in G$, $g \in H$

$D_n = \{R_0, R_1, \dots, R_{n-1}, S_0, S_1, \dots, S_{n-1}\}$, the group of symmetries of a regular n-gon

$\mathbb{Z}_n = D_n / \{S_0, S_1, \dots, S_{n-1}\} \subset D_n$

d) What are conjugacy classes? Determine all conjugacy classes of D_4 and D_5. What about general D_n?

conjugacy class: $C_g = [g] = \{g' \in G \mid \tilde{g}\, g\, \tilde{g}^{-1} = g\}$

$D_4 = \{R_0, R_1, R_2, R_3, S_0, S_1, S_2, S_3\}$

$[R_0] = \{R_0, R_1, R_2, R_3, S_0, S_1, S_2, S_3\}$

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Exercise 2 (Finite groups)

a) Let G be a finite group with a prime number of elements. What can you say about G?

By Lagrange's theorem, G can only have the two trivial subgroups $\{e\}$ and G. Since we can always construct orbits Γ_g for all elements $g \in G$, which by the closure axiom, are subgroups of G, we have $\Gamma_e = \{e\}$ or $\Gamma_g = G$. Since Γ_g is isomorphic to $\mathbb{Z}_{|G|}$, $\Gamma_g \cong \mathbb{Z}_{|G|}$, we have that G is isomorphic to $\mathbb{Z}_{|G|}$, $G \cong \mathbb{Z}_{|G|}$.

b) Write the multiplication table of $\mathbb{Z}_3$ and the respective embedding $\mathbb{Z}_3 \hookrightarrow S_3$ into the symmetric group.

$\mathbb{Z}_3$	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

$\mathbb{Z}_3 \overset{\pi}{\hookrightarrow} S_3$

$\pi_0 : 0 \longrightarrow \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$

$\pi_1 : 1 \longrightarrow \begin{pmatrix} 0 & 1 & 2 \\ 1 & 2 & 0 \end{pmatrix}$

$\pi_2 : 2 \longrightarrow \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{pmatrix}$

Exercise 3 (Symmetric group)

a) Write the permutation $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 5 & 2 & 4 & 6 \end{pmatrix}$ as product of non-intersecting cycles.

$\pi = (1\ 3\ 5\ 4\ 2)(6)$

b) What is the order of S_3? How many conjugacy classes does S_3 have? Determine their sizes, and give a representation of each one.

$|S_3| = 3! = 6$. S_3 has $p(3) = 3$ conjugacy classes, where $p(n)$ is the number of partitions of n. Conjugacy class sizes can be calculated using $|C_{(k_1, \dots, k_n)}| = \frac{n!}{\prod_i i^{k_i} k_i!}$, where the k_i give the number of permutation cycles of length i.

Considering the different partitions of n, we get

$\lambda = (3,0,0)$, $k = (3,0,0)$ $\qquad c_{(3,0,0)} = \frac{3!}{1^3 3! \cdot 2^0 0! \cdot 3^0 0!} = 1$

$\lambda = (2,1,0)$, $k = (1,1,0)$ $\qquad c_{(1,1,0)} = \frac{3!}{1^1 1! \cdot 2^1 1! \cdot 3^0 1!} = 3$

$\lambda = (1,1,1)$, $k = (0,0,1)$ $\qquad c_{(0,0,1)} = \frac{3!}{1^0 \cdot 0! \cdot 2^0 \cdot 0! \, 3^1 1!} = 2$

Representatives: $\lambda = (3,0,0)$: $e = (1)(2)(3)$

$\lambda = (2,1,0)$: $(12)(3)$

$\lambda = (1,1,1)$: (123)

c) Show that $H = \{e, (1,2)\} \subset S_3$ is a subgroup. Is it normal?

Closure of H: $ee = e \in H$, $e(1,2) = (1,2) \in H$ $\quad (1,2)e = (1,2) \in H$

$(1,2)(1,2) = e \in H$

Associativity: Is inherited from S_3, since $e, (1,2) \in S_3$.

Identity: e ✓

Inverse: $ee = e$, $(1,2)(1,2) = e$ ✓

All four group axioms are met. Therefore $H \subset S_3$ is a subgroup.

Since H cannot be formed by conjugacy classes of S_3, it is not normal in S_3.

Exercise 4 (Commutator subgroup)

Let G be a group. The commutator of two elements $g, h \in G$ is defined by $[g,h] := g^{-1} h^{-1} g h \in G$. The commutator subgroup of G is the group generated by all commutators in G:

$$[G,G] = \{[g_1,h_1], \dots, [g_n,h_n] \mid n \in \mathbb{N}, g_i, h_i \in G\}$$

a) Show that the commutator subgroup is indeed a subgroup $[G,G] \subset G$.

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We use the condition that if for all $[g,h], [k,l] \in [G,G]$, $[g,h][k,l]^{-1} \in [G,G]$, where $g,h,k,l \in G$, then $[G,G] < G$ is a subgroup.

We have $[k,l]^{-1} = (k^{-1}l^{-1}kl)^{-1} = l^{-1}k^{-1}lk = [l,k]$ and therefore $[g,h][k,l]^{-1} = [g,h][l,k] \in [G,G]$ by definition of $[G,G]$.

b) Show that it is a normal subgroup $[G,G] \triangleleft G$.

$[G,G] \triangleleft G$ is a normal subgroup $\iff \forall g \in G: g[G,G]g^{-1} = [G,G]$

We prove the above equality by showing that each is a subgroup of the other.

$g[G,G]g^{-1} \subset [G,G]$:

$\forall g \in G, [k,l] \in [G,G]$, we have $[k,l]^{-1} g [k,l] g^{-1} = [[k,l], g^{-1}] \in [G,G]$.

From this, it follows that, since $[G,G] < G$ is a subgroup fulfilling closure w.r.t. the group action,

$$[k,l][k,l]^{-1} g [k,l] g^{-1} = g[k,l]g^{-1} \in [G,G]$$

$g[G,G]g^{-1} \supset [G,G]$:

$\forall [k,l] \in [G,G] \ \exists m,n \in G: \ gmg^{-1} = k \ \wedge \ gng^{-1} = l$,

where we used that conjugation is a homomorphism on G

$$\implies [k,l] = [gmg^{-1}, gng^{-1}] = (gmg^{-1})^{-1}(gng^{-1})^{-1} gmg^{-1} gng^{-1}$$
$$= gm^{-1}g^{-1} gn^{-1}g^{-1} gmg^{-1}gng^{-1} = gm^{-1}n^{-1}mng^{-1}$$
$$= g[m,n]g^{-1} \in g[G,G]g^{-1}$$

Therefore, $[G,G] \triangleleft G$.

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Exercise 5 (Group actions)

An action of a group G on a set X is a group homomorphism $\varphi: G \to \mathrm{Bij}(X)$ from G into the group $\mathrm{Bij}(X)$ of bijections from X to X. One often uses the abbreviated notation $gx := \varphi(g)(x)$. For any $x \in X$ its orbit in G is defined by

$$\mathrm{Orb}_G(x) = Gx = \{gx \mid g \in G\} \subset X.$$

The stabilizer of $x \in X$ is defined by

$$\mathrm{Stab}_G(x) = \{g \in G \mid gx = x\} \subset G.$$

a) Show that the stabilizer is a subgroup of G.

For this, we again resort to the 'one-step' subgroup test, i.e. we show that $\mathrm{Stab}_G(x) \subset G$ is a sub<u>set</u> of G and that $\mathrm{Stab}_G(x) \neq \emptyset$ is not empty (both of which is fulfilled by definition) and that $\forall\, a, b \in \mathrm{Stab}_G(x)$, we have $ab^{-1} \in \mathrm{Stab}_G(x)$, where always $x \in X$. For such a and b, we can write

$$b^{-1}x = b^{-1}(bx) \overset{\varphi}{=} (b^{-1}b)\,x = x \quad \text{where we used that } \varphi \text{ is homomorphism}$$

$$\Longrightarrow (ab^{-1})x = a\,(b^{-1}x) = ax = x$$

$$\Longrightarrow ab^{-1} \in \mathrm{Stab}_G(x) \text{ and } \mathrm{Stab}_G(x) \leq G$$

b) Show that stabilizers of elements $y, y' \in X$ which lie in the same G-orbit (i.e. $y, y' \in \mathrm{Orb}_G(x)$ for some $x \in X$) are conjugate.

Since $y, y' \in \mathrm{Orb}_G(x)$, $\exists\, a, b \in G$: $ax = y$, $bx = y'$. G being a group, we can invert to get $a^{-1}y = x = b^{-1}y'$.

For any $g \in \mathrm{Stab}_G(y)$, we have $gy = y \Leftrightarrow g(ax) = ax \Leftrightarrow a^{-1}g(ax) = x$.

Inserting $x = b^{-1}y'$ gives $a^{-1}g(ab^{-1}y') = b^{-1}y' \Longleftrightarrow (ba^{-1})\,g\,(ab^{-1})y' = y'$, from which we deduce that $\underbrace{(ba^{-1})}_{h}\,g\,\underbrace{(ab^{-1})}_{h^{-1}} =: hgh^{-1} \in \mathrm{Stab}_G(y')$.

Since this holds for all $g \in \mathrm{Stab}_G(y)$, we found $h\,\mathrm{Stab}_G(y)\,h^{-1} = \mathrm{Stab}_G(y')$.

c) Show that the map $Gx \longrightarrow G/\mathrm{Stab}_G(x)$, which sends $gx \longmapsto g\,\mathrm{Stab}_G(x)$ is well defined (i.e. $gx = hx \Rightarrow g\,\mathrm{Stab}_G(x) = h\,\mathrm{Stab}_G(x)$), and is a bijection.