String Theory

Solution to Assignment 1

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1 Differential Geometry for General Relativity

Consider the line element of a 2-sphere of radius a,

$$\mathrm{d}s^2 = g_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu = a^2[\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2]. \tag{1}$$

The metric $g_{\mu\nu}$ encodes all information on the geometry of a manifold. From it one can determine all those geometric quantities that are relevant for general relativity, namely

The metric Choosing $x^1 = \theta$ and $x^2 = \phi$, read off the matrix $g_{\mu\nu}$.

The Christoffel symbols The Christoffel symbols are defined as

$$\Gamma^\lambda{}_{\mu\nu} = \frac{1}{2} g^{\lambda\kappa}\left(\frac{\partial g_{\mu\kappa}}{\partial x^\nu} + \frac{\partial g_{\nu\kappa}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\kappa}\right). \tag{2}$$

They enter the covariant derivatives $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\lambda{}_{\mu\nu} V^\lambda$, where the correction term with the Christoffel symbol ensures that the covariant derivative indeed transforms covariantly under arbitrary coordinate transformations $x^\mu \to x'^\mu(x^\nu)$, i.e.

$$\nabla_\mu V^\nu = \partial_\mu V^\nu \to (\nabla_\mu V^\nu = \partial_\mu V^\nu)' = \frac{\partial x^\lambda}{\partial x'^\mu}\frac{\partial x'^\nu}{\partial x^\rho}\nabla_\lambda V^\rho, \tag{3}$$

without second derivatives in the coordinates. Compute the non-vanishing Christoffel symbols for the 2-sphere (Hint: $\Gamma^\kappa{}_{\lambda\mu} = \Gamma^\kappa{}_{\mu\lambda}$).

The Riemann tensor The Riemann curvature tensor has the form

$$R^\kappa{}_{\lambda\mu\nu} = \partial_\mu \Gamma^\kappa{}_{\lambda\nu} - \partial_\nu \Gamma^\kappa{}_{\lambda\mu} + \Gamma^\eta{}_{\lambda\nu}\Gamma^\kappa{}_{\eta\mu} - \Gamma^\eta{}_{\lambda\mu}\Gamma^\kappa{}_{\nu\eta}. \tag{4}$$

Calculate the non-vanishing components of the Riemann tensor for the 2-sphere (Hint: $R^\kappa{}_{\lambda\mu\nu} = -R^\kappa{}_{\lambda\nu\mu}$).

Remark: The Riemann tensor measures the curvature of a space, for instance by quantifying the non-commutativity of the covariant derivatives,

$$[\nabla_\mu, \nabla_\nu]V^\kappa = R^\kappa{}_{\lambda\mu\nu} V^\lambda. \tag{5}$$

A space with vanishing Riemann tensor is flat, i.e. the metric can be brought to the standard Minkowskian (or Euclidean) form by means of a coordinate transformation.

The Ricci tensor The Ricci tensor is defined as

$$R_{\mu\nu} = R^{\kappa}{}_{\mu\kappa\nu}. \tag{6}$$

Calculate the Ricci tensor for the 2-sphere.

The scalar curvature The scalar curvature is given by

$$\mathcal{R} = g^{\mu\nu} R_{\mu\nu}. \tag{7}$$

Calculate the scalar curvature of the 2-sphere. How does it behave in the limit $a \to \infty$? Interpret this behavior.

The Einstein tensor The Einstein tensor appears in the field equation of general relativity and it relates the curvature of space-time to the matter distribution,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{8}$$

where G denotes Newton's constant, $T_{\mu\nu}$ is the energy-momentum tensor, and $G_{\mu\nu}$ denotes the Einstein tensor,

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}\mathcal{R} g_{\mu\nu}. \tag{9}$$

Calculate the Einstein tensor for the 2-sphere.

The metric For $x^1 = \theta$ and $x^2 = \phi$, the equation

$$g_{\mu\nu} \mathrm{d}x^\mu \mathrm{d}x^\nu = a^2[\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2] \tag{10}$$

implies

$$\boldsymbol{g} = a^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2(\theta) \end{pmatrix}, \qquad \text{and hence} \quad \boldsymbol{g}^{-1} = \frac{1}{a^2} \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\sin^2(\theta)} \end{pmatrix}. \tag{11}$$

The Christoffel symbols Since the Christoffel symbols carry three coordinate indices and we have $d = 2$ dimensions (θ, ϕ), there are $d^3 = 8$ Christoffel symbols in total. However, due to the symmetry in the lower two indices, those with lower indices 12 and 21 are equal both for an upper index of 1 and 2, so only $8 - 2 = 6$ of those symbols are independent. We calculate each of those in turn. Since the only nonvanishing metric derivative is $\frac{\partial g_{22}}{\partial x^1} = 2\sin(\theta)\cos(\theta)$, all but $\Gamma^1{}_{22}$, $\Gamma^2{}_{12}$, and $\Gamma^2{}_{21}$ can immediately be seen to vanish:

$$\Gamma^1{}_{11} = \frac{1}{2} g^{1\kappa} \left(\frac{\partial g_{1\kappa}}{\partial x^1} + \frac{\partial g_{1\kappa}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^\kappa} \right) = 0, \tag{12}$$

$$\Gamma^1{}_{12} = \frac{1}{2} g^{1\kappa} \left(\frac{\partial g_{1\kappa}}{\partial x^2} + \frac{\partial g_{2\kappa}}{\partial x^1} - \frac{\partial g_{12}}{\partial x^\kappa} \right) = \frac{1}{2} g^{12} \frac{\partial g_{22}}{\partial x^1} = 0 = \Gamma^1{}_{21}, \tag{13}$$

$$\begin{aligned} \Gamma^1{}_{22} &= \frac{1}{2} g^{1\kappa} \left(\frac{\partial g_{2\kappa}}{\partial x^2} + \frac{\partial g_{2\kappa}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^\kappa} \right) \\ &= -\frac{1}{2} g^{11} \frac{\partial g_{22}}{\partial x^1} = -\frac{1}{2a^2} \frac{\partial[a^2 \sin^2(\theta)]}{\partial \theta} = -\sin(\theta)\cos(\theta), \end{aligned} \tag{14}$$

$$\Gamma^2{}_{11} = \frac{1}{2} g^{2\kappa} \left(\frac{\partial g_{1\kappa}}{\partial x^1} + \frac{\partial g_{1\kappa}}{\partial x^1} - \frac{\partial g_{11}}{\partial x^\kappa} \right) = 0, \tag{15}$$

$$
\begin{aligned}
\Gamma^2{}_{12} &= \frac{1}{2} g^{2\kappa}\left(\frac{\partial g_{1\kappa}}{\partial x^2} + \frac{\partial g_{2\kappa}}{\partial x^1} - \frac{\partial g_{12}}{\partial x^\kappa}\right) \\
&= \frac{1}{2} g^{22}\frac{\partial g_{22}}{\partial x^1} = \frac{1}{2a^2\sin^2(\theta)}\frac{\partial[a^2\sin^2(\theta)]}{\partial\theta} = \cot(\theta) = \Gamma^2{}_{21},
\end{aligned} \tag{16}
$$

$$
\Gamma^2{}_{22} = \frac{1}{2} g^{2\kappa}\left(\frac{\partial g_{2\kappa}}{\partial x^2} + \frac{\partial g_{2\kappa}}{\partial x^2} - \frac{\partial g_{22}}{\partial x^\kappa}\right) = 0. \tag{17}
$$

The Riemann tensor The Riemann tensor has four indices. So for $d = 2$ dimensions, the tensor contains a total of $d^4 = 16$ components. Due to the antisymmetry $R^\kappa{}_{\lambda\mu\nu} = -R^\kappa{}_{\lambda\nu\mu}$ of the last two indices, there is only $\frac{d}{2}(d-1) = 1$ independent combination of those two indices, leaving the Riemann tensor with $d^2 \cdot \frac{d}{2}(d-1) = 4$ independent components. In particular, all eight entries where the last two indices are equal must be zero, i.e.

$$
R^1{}_{111} = R^1{}_{211} = R^2{}_{111} = R^2{}_{211} = 0, \tag{18}
$$

$$
R^1{}_{122} = R^1{}_{222} = R^2{}_{122} = R^2{}_{222} = 0. \tag{19}
$$

The remaining eight components are potentially nonzero, but form four pairs of two whose members differ only in sign. These we calculate by hand:

$$
R^1{}_{112} = -R^1{}_{121} = \partial_1\Gamma^1{}_{12} - \partial_2\Gamma^1{}_{11} + \Gamma^\eta{}_{12}\Gamma^1{}_{\eta1} - \Gamma^\eta{}_{11}\Gamma^1{}_{2\eta} = 0, \tag{20}
$$

$$
\begin{aligned}
R^1{}_{212} = -R^1{}_{221} &= \partial_1\Gamma^1{}_{22} - \partial_2\Gamma^1{}_{22} + \Gamma^\eta{}_{12}\Gamma^1{}_{\eta1} - \Gamma^\eta{}_{21}\Gamma^1{}_{2\eta} = \partial_1\Gamma^1{}_{22} - \Gamma^2{}_{21}\Gamma^1{}_{22} \\
&= \partial_\theta[-\sin(\theta)\cos(\theta)] - \cot(\theta)\cdot[-\sin(\theta)\cos(\theta)] \\
&= -\cos^2(\theta) + \sin^2(\theta) + \cos^2(\theta) = \sin^2(\theta),
\end{aligned} \tag{21}
$$

$$
\begin{aligned}
R^2{}_{112} = -R^2{}_{121} &= \partial_1\Gamma^2{}_{12} - \partial_2\Gamma^2{}_{11} + \Gamma^\eta{}_{12}\Gamma^2{}_{\eta1} - \Gamma^\eta{}_{11}\Gamma^2{}_{2\eta} = \partial_1\Gamma^2{}_{12} + \Gamma^2{}_{12}\Gamma^2{}_{21} \\
&= \partial_\theta\cot(\theta) + \cot(\theta)\cdot\cot(\theta) = -1 - \cot^2(\theta) + \cot^2(\theta) = -1,
\end{aligned} \tag{22}
$$

$$
R^2{}_{212} = -R^2{}_{221} = \partial_1\Gamma^2{}_{22} - \partial_2\Gamma^2{}_{21} + \Gamma^\eta{}_{22}\Gamma^2{}_{\eta1} - \Gamma^\eta{}_{21}\Gamma^2{}_{2\eta} = 0. \tag{23}
$$

We found four nonvanishing components. The remark given in the exercise that a space with vanishing Riemann tensor is flat is in fact an "iff"-statement, i.e. a nonvanishing Riemann tensor implies that space is curved. We have therefore proven the unremarkable statement that the 2-sphere is curved.

The Ricci tensor The $d^2 = 4$ components of the Ricci tensor of the 2-sphere are given by

$$
R_{11} = R^\kappa{}_{1\kappa1} = R^1{}_{111} + R^2{}_{121} = 1, \tag{24}
$$

$$
R_{12} = R^\kappa{}_{1\kappa2} = R^1{}_{112} + R^2{}_{122} = 0, \tag{25}
$$

$$
R_{21} = R^\kappa{}_{2\kappa1} = R^1{}_{211} + R^2{}_{221} = 0, \tag{26}
$$

$$
R_{22} = R^\kappa{}_{2\kappa2} = R^1{}_{212} + R^2{}_{222} = \sin^2(\theta). \tag{27}
$$

The scalar curvature In the case of the 2-sphere, $\mathcal{R}$ takes the very simple and memorable form,

$$
\mathcal{R} = g^{\mu\nu}R_{\mu\nu} = g^{11}R_{11} + g^{22}R_{22} = \frac{1}{a^2}\cdot 1 + \frac{1}{a^2\sin^2(\theta)}\cdot\sin^2(\theta) = \frac{2}{a^2}. \tag{28}
$$

As expected, in the limit $a \to \infty$, we have

$$
\lim_{a\to\infty}\mathcal{R} = 0, \tag{29}
$$

i.e. a sphere of infinite radius has vanishing curvature.

The Einstein tensor We give the Einstein tensor of the 2-sphere not component-wise, but in covariant matrix form:

$$
\boldsymbol{G} = \boldsymbol{R} - \frac{1}{2}\mathcal{R}\boldsymbol{g} = \begin{pmatrix}1 & 0\\ 0 & \sin^2(\theta)\end{pmatrix} - \frac{1}{2}\frac{2}{a^2}a^2\begin{pmatrix}1 & 0\\ 0 & \sin^2(\theta)\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}. \tag{30}
$$

According to the Einstein equation,

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}, \tag{31}$$

a vanishing Einstein tensor requires a trivial matter distribution, $T_{\mu\nu} = 0$.

2 Transformation of tensors and tensor densities

Consider the coordinate change

$$x^\mu \to x'^\mu \equiv x^{\mu'}. \tag{32}$$

The associated transformation matrix and its inverse are

$$P^\mu{}_{\nu'} = \frac{\partial x^\mu}{\partial x'^\nu}, \qquad \text{and} \qquad P^{\mu'}{}_\nu = \frac{\partial x'^\mu}{\partial x^\nu}, \tag{33}$$

respectively. Recall that a tensor of type, say $T^\mu{}_\nu$ transforms under eq. (32) as

$$T^\mu{}_\nu \to T^{\mu'}{}_{\nu'} = P^{\mu'}{}_\alpha P^\beta{}_{\nu'} T^\alpha{}_\beta . \tag{34}$$

A tensor density $\tilde{T}^\mu{}_\nu$ of weight w is defined by the transformation behavior

$$\tilde{T}^\mu{}_\nu \to \tilde{T}^{\mu'}{}_{\nu'} = J^w P^{\mu'}{}_\alpha P^\beta{}_{\nu'} \tilde{T}^\alpha{}_\beta . \tag{35}$$

(and obvious generalisations for general types of tensor densities), where $J = \det(\boldsymbol{P})$.

a) Given the tensor $S_{\mu\nu}$, convince yourself that $\sqrt{\det(\boldsymbol{S})}$ is a scalar density of weight 1.

b) Consider now fields of tensors and tensor densities, e.g. $T^\mu{}_\nu(x)$. Locally, i.e. infinitesimally, the transformation of eq. (32) can be parametrized as $x'^\mu = x^\mu - \epsilon^\mu(x)$. Show the following infinitesimal variations for a scalar field $\Phi(x)$, the metric $g_{\mu\nu}(x)$ and the associated metric density $\sqrt{-\det(\boldsymbol{g})}$:

   i) $\delta\Phi = \epsilon^\mu \partial_\mu \Phi$,

   ii) $\delta g_{\mu\nu} = \epsilon^\lambda \partial_\lambda g_{\mu\nu} + (\partial_\mu \epsilon^\lambda) g_{\lambda\nu} + (\partial_\nu \epsilon^\lambda) g_{\mu\lambda} = \nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu$,

   iii) $\delta\sqrt{-\det(\boldsymbol{g})} = \partial_\lambda[\epsilon^\lambda \sqrt{-\det(\boldsymbol{g})}]$,

   where the second equality in ii) is true for the metric connection satisfying $\nabla_\lambda g_{\mu\nu} = 0$.

   Hint: For a scalar field the transformed object is defined via the relation $\Phi'(x') = \Phi(x)$.

a) Since $S_{\mu\nu}$ is said to be a tensor, we know it transforms as

$$S_{\mu\nu} \to S_{\mu'\nu'} = P^\alpha{}_{\mu'} P^\beta{}_{\nu'} S_{\alpha\beta}. \tag{36}$$

or in matrix notation

$$\boldsymbol{S} \to \boldsymbol{S}' = \boldsymbol{P}^2_{x\to x'} \boldsymbol{S}. \tag{37}$$

Using that the determinant of a product of matrices is the product of the determinants, we have

$$\begin{aligned}\sqrt{\det(\boldsymbol{S})} \to \sqrt{\det(\boldsymbol{S}')} = \sqrt{\det(\boldsymbol{P}^2_{x\to x'}\boldsymbol{S})} &= \sqrt{[\det(\boldsymbol{P}_{x\to x'})]^2 \det(\boldsymbol{S})} \\ = \det(\boldsymbol{P}_{x\to x'})\sqrt{\det(\boldsymbol{S})} &= J^1_{x\to x'}\sqrt{\det(\boldsymbol{S})},\end{aligned} \tag{38}$$

and thus $\sqrt{\det(\boldsymbol{S})}$ is a tensor density of weight 1.

b) We now derive the infinitesimal transformation behavior of a scalar field as well as the metric and the square root of its negated determinant under the transformation $x'^\mu = x^\mu - \epsilon^\mu(x')$.

i) Using $x^\mu = x'^\mu + \epsilon^\mu(x')$, the variation of a **scalar field** follows from the hint that $\Phi'(x') = \Phi(x)$ together with a simple Taylor expansion of $\Phi(x)$ around x',

$$\begin{aligned}\Phi(x) \to \Phi'(x') &\overset{!}{=} \Phi(x) = \Phi(x' + \epsilon(x')) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{\partial^n \Phi(x' + \epsilon(x'))}{\partial^n (x'^\mu + \epsilon^\mu(x'))} \right|_{x'^\mu + \epsilon^\mu(x') = x'^\mu} \Big(x'^\mu + \epsilon^\mu(x') - x'^\mu\Big)^n \\ &= \Phi(x') + \epsilon^\mu(x')\partial_{\mu'}\Phi(x') + \mathcal{O}[\epsilon^2(x')].\end{aligned} \tag{39}$$

From this transformation law it follows that

$$\Phi'(x) = \Phi(x) + \epsilon^\mu(x)\partial_\mu \Phi(x) + \mathcal{O}[\epsilon^2(x)], \tag{40}$$

and hence the variation $\delta\Phi(x)$ of the scalar field $\Phi(x)$ given by the difference of the transformed and the original field reads

$$\delta\Phi(x) = \Phi'(x) + \Phi(x) = \epsilon^\mu(x)\partial_\mu \Phi(x) + \mathcal{O}[\epsilon^2(x)]. \tag{41}$$

ii) For the **metric** $g_{\mu\nu}(x)$, we know that it strictly follows the transformational behavior (34) of a tensor (field). Therefore,

$$g_{\mu\nu}(x) \to g_{\mu'\nu'}(x') = P^\alpha{}_{\mu'}\, P^\beta{}_{\nu'}\, g_{\alpha\beta}(x). \tag{42}$$

Now, all we have to do is expand the expression on the right to first order in $\epsilon^\mu(x)$. This can be done by inserting our transformation $x^\mu = x'^\mu + \epsilon^\mu(x')$ into the definition of the transformation matrix $P^\mu{}_{\nu'}$

$$P^\alpha{}_{\mu'} = \frac{\partial x^\alpha}{\partial x'^\mu} = \frac{\partial x'^\alpha + \epsilon^\alpha(x')}{\partial x'^\mu} = \delta^\alpha{}_\mu + \partial_{\mu'}\epsilon^\alpha(x'). \tag{43}$$

For the metric $g_{\alpha\beta}(x)$, we do a Taylor expansion,

$$\begin{aligned}g_{\alpha\beta}(x) &= g_{\alpha\beta}(x' + \epsilon(x')) \\ &= \sum_{n=0}^{\infty} \frac{1}{n!} \left. \frac{\partial^n g_{\alpha\beta}(x' + \epsilon(x'))}{\partial^n (x'^\gamma + \epsilon^\gamma(x'))} \right|_{x'^\gamma + \epsilon^\gamma(x') = x'^\gamma} \Big(x'^\gamma + \epsilon^\gamma(x') - x'^\gamma\Big)^n \\ &= g_{\alpha\beta}(x') + \epsilon^\gamma(x')\partial_{\gamma'} g_{\alpha\beta}(x') + \mathcal{O}[\epsilon^2(x')].\end{aligned}$$

Inserting eq. (43) and part ii) into eq. (42), we get

$$\begin{aligned}g_{\mu'\nu'}(x') &= [\delta^\alpha{}_\mu + \partial_{\mu'}\epsilon^\alpha(x')][\delta^\beta{}_\nu + \partial_{\nu'}\epsilon^\beta(x')]\Big[g_{\alpha\beta}(x') + \epsilon^\gamma(x')\partial_{\gamma'} g_{\alpha\beta}(x') + \mathcal{O}[\epsilon^2(x')]\Big] \\ &= \delta^\alpha{}_\mu \delta^\beta{}_\nu g_{\alpha\beta}(x') + [\partial_{\mu'}\epsilon^\alpha(x')]\delta^\beta{}_\nu g_{\alpha\beta}(x') + \delta^\alpha{}_\mu[\partial_{\nu'}\epsilon^\beta(x')]g_{\alpha\beta}(x') \\ &\quad + \delta^\alpha{}_\mu \delta^\beta{}_\nu \epsilon^\gamma(x')\partial_{\gamma'} g_{\alpha\beta}(x') + \mathcal{O}[\epsilon^2(x')] \\ &= g_{\mu\nu}(x') + g_{\alpha\nu}(x')\partial_{\mu'}\epsilon^\alpha(x') + g_{\mu\beta}(x')\partial_{\nu'}\epsilon^\beta(x') + \epsilon^\gamma(x')\partial_{\gamma'} g_{\mu\nu}(x') + \mathcal{O}[\epsilon^2(x')].\end{aligned}$$

From this we infer that

$$g_{\mu'\nu'}(x) = g_{\mu\nu}(x) + g_{\alpha\nu}(x)\partial_\mu \epsilon^\alpha(x) + g_{\mu\beta}(x)\partial_\nu \epsilon^\beta(x) + \epsilon^\gamma(x)\partial_\gamma g_{\mu\nu}(x) + \mathcal{O}[\epsilon^2(x)], \tag{44}$$

and hence

$$\begin{aligned}\delta g_{\mu\nu}(x) &= g_{\mu'\nu'}(x) - g_{\mu\nu}(x) \\ &= g_{\alpha\nu}(x)\partial_\mu \epsilon^\alpha(x) + g_{\mu\beta}(x)\partial_\nu \epsilon^\beta(x) + \epsilon^\gamma(x)\partial_\gamma g_{\mu\nu}(x) + \mathcal{O}[\epsilon^2(x)].\end{aligned} \tag{45}$$

To see that the second equality in ii) holds, we simply calculate

$$
\begin{aligned}
\nabla_\mu \epsilon_\nu + \nabla_\nu \epsilon_\mu = \nabla_\mu (g_{\alpha\nu}\epsilon^\alpha) + \nabla_\nu (g_{\mu\beta}\epsilon^\beta) &= g_{\alpha\nu}\nabla_\mu \epsilon^\alpha + g_{\mu\beta}\nabla_\nu \epsilon^\beta + \epsilon^\gamma \underbrace{\nabla_\gamma g_{\mu\nu}}_{0} \\
&= g_{\alpha\nu}\left(\partial_\mu \epsilon^\alpha + \Gamma^\alpha{}_{\mu\delta}\epsilon^\delta\right) + g_{\mu\beta}\left(\partial_\nu \epsilon^\beta + \Gamma^\beta{}_{\nu\zeta}\epsilon^\zeta\right) \\
&\quad + \epsilon^\gamma \left(\partial_\gamma g_{\mu\nu} - \Gamma^\alpha{}_{\mu\gamma} g_{\alpha\nu} - \Gamma^\beta{}_{\nu\gamma} g_{\mu\beta}\right) \\
&= g_{\alpha\nu}\partial_\mu \epsilon^\alpha + g_{\mu\beta}\partial_\nu \epsilon^\beta + \epsilon^\gamma \partial_\gamma g_{\mu\nu} \\
&\quad + g_{\alpha\nu}\Gamma^\alpha{}_{\mu\delta}\epsilon^\delta - \epsilon^\gamma \Gamma^\alpha{}_{\mu\gamma} g_{\alpha\nu} + g_{\mu\beta}\Gamma^\beta{}_{\nu\zeta}\epsilon^\zeta - \epsilon^\gamma \Gamma^\beta{}_{\nu\gamma} g_{\mu\beta} \\
&= g_{\alpha\nu}\partial_\mu \epsilon^\alpha + g_{\mu\beta}\partial_\nu \epsilon^\beta + \epsilon^\gamma \partial_\gamma g_{\mu\nu} = \delta g_{\mu\nu} - \mathcal{O}[\epsilon^2].
\end{aligned}
\tag{46}
$$

iii) By part a), we have

$$
\sqrt{-\det[\boldsymbol{g}(x)]} \to \sqrt{-\det[\boldsymbol{g}'(x')]} = J_{x\to x'}\sqrt{-\det[\boldsymbol{g}(x)]} = J_{x\to x'}g(x), \tag{47}
$$

where to save on writing, we introduced the shorthand notation $\sqrt{-\det[\boldsymbol{g}(x)]} = g(x)$. To find the infinitesimal variation $\delta\sqrt{-\det[\boldsymbol{g}(x)]} = \delta g(x)$, we expand eq. (47) to first order in $\epsilon(x)$,

$$
\begin{aligned}
g'(x') = J_{x\to x'}g(x) &= \det(\boldsymbol{P}_{x\to x'})g(x'+\epsilon(x')) \\
&= \underbrace{\det[\mathbf{1} + \boldsymbol{\partial}_{x'}\boldsymbol{\epsilon}(x')]}_{1+\mathrm{Tr}[\boldsymbol{\partial}_{x'}\boldsymbol{\epsilon}(x')]+\mathcal{O}[\epsilon^2(x')]}\Big(g(x') + \epsilon^\gamma(x')\partial_{\gamma'}g(x') + \mathcal{O}[\epsilon^2(x')]\Big) \\
&= g(x') + [\partial_{\mu'}\epsilon^\mu(x')]g(x') + \epsilon^\mu(x')\partial_{\mu'}g(x') + \mathcal{O}[\epsilon^2(x')] \\
&= g(x') + \partial_{\mu'}[\epsilon^\mu(x')g(x')] + \mathcal{O}[\epsilon^2(x')],
\end{aligned}
\tag{48}
$$

where to get to the second line we used $\boldsymbol{P}_{x\to x'} = \mathbf{1} + \boldsymbol{\partial}_{x'}\boldsymbol{\epsilon}(x')$, i.e. eq. (43) in matrix form. From eq. (48), we infer

$$
g'(x) = g(x) + \partial_\mu[\epsilon^\mu(x)g(x)] + \mathcal{O}[\epsilon^2(x)], \tag{49}
$$

from which in turn it follows that

$$
\delta g(x) = g'(x) - g(x) = \partial_\mu[\epsilon^\mu(x)g(x)] + \mathcal{O}[\epsilon^2(x)], \tag{50}
$$

or in unshortened notation,

$$
\delta\sqrt{-\det[\boldsymbol{g}(x)]} = \partial_\mu[\epsilon^\mu(x)\sqrt{-\det[\boldsymbol{g}(x)]}] + \mathcal{O}[\epsilon^2(x)]. \tag{51}
$$

3 Action Principle

a) Consider a field $\varphi(x)$ and an action in d spacetime dimensions with Minkowskian signature $(-1, +1, \ldots, +1)$ of the form

$$
S[\varphi] = \int \mathrm{d}^d x\, \mathcal{L}(\varphi(x), \partial_\mu\varphi(x)). \tag{52}
$$

The simplest example is the action for the free scalar field

$$
S[\varphi] = -\frac{1}{2}\int \mathrm{d}^d x \left(\partial_\mu\varphi\partial^\mu\varphi + m^2\varphi^2\right). \tag{53}
$$

The variation of the general action (52) is defined as

$$
\delta S[\varphi] = \int \mathrm{d}^d x \left(\frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta(\partial_\mu\varphi)\right). \tag{54}
$$

Using integration by parts and neglecting boundary terms show that the action principle $\delta S[\varphi] = 0$ implies the Euler Lagrange equations

$$\partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = \frac{\partial \mathcal{L}}{\partial \varphi}. \tag{55}$$

b) Use this result to derive the Klein-Gordon equation as the Euler-Lagrange equation of the free scalar field action (53).

a) Using partial integration on the second term in the variation of the action (54), $\delta S[\varphi]$ becomes

$$\delta S[\varphi] = \underbrace{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta\varphi \Big|_{\partial \mathbb{R}^d}}_{0} + \int \mathrm{d}^d x \left(\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) \delta\varphi. \tag{56}$$

Since we assume we are working with localized *physical* systems, we take the boundary terms to vanish at spatial and temporal infinity, i.e. the term in front can be disregarded.

At this point, we resort to Hamilton's principle of a stationary action, $\delta S[\varphi] = 0$, which we require to hold for all possible variations $\delta\varphi$ of $\varphi(x)$. $\delta S[\varphi] = 0$ can hence only be true in general if the integrand itself vanishes. We thus arrive at the renowned Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = 0. \tag{57}$$

b) Comparing eqs. (52) and (53), we see that the Lagrangian $\mathcal{L}(\varphi(x), \partial_\mu \varphi(x))$ of the free scalar field is given by

$$\mathcal{L}(\varphi(x), \partial_\mu \varphi(x)) = -\frac{1}{2}\Big(\partial_\mu \varphi \partial^\mu \varphi + m^2 \varphi^2 \Big). \tag{58}$$

Inserting this expression into eq. (57), we obtain as equation of motion,

$$-\, m^2 \varphi + \partial_\mu \partial^\mu \varphi = 0, \tag{59}$$

which written in a more familiar form reads

$$\left(\Box_x - m^2 \right) \varphi(x) = 0. \tag{60}$$