09.07.2015

Quantum Field Theory II – Final Exam by Tillmann Plehn

Problem 1: One-loop calculations in QED

1. QED with massless fermion field $\Psi(x)$ and massless gauge field $A_\mu(x)$

$$\mathcal{L} = -\frac{1}{4} F_{0,\mu\nu} F_0^{\mu\nu} + i\overline{\Psi}_0 \not{D}_0 \Psi_0, \qquad D_0^\mu = \partial^\mu + ie_0 A_0^\mu$$

We use the Feynman rules to derive the one-loop corrections to the fermion propagator $i\mathcal{M}[f(p) \to f(p)]$ of a fermion f with mom. p. To apply renormalized perturbation theory, we split the bare Lagrangian into renormalized quantities, e.g. $\Psi = Z_\Psi^{-\frac{1}{2}} \Psi_0$ and $A^\mu = Z_A^{-\frac{1}{2}} A_0^\mu$, and counter-terms as follows

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + i\overline{\Psi}\not{\partial}\Psi - e A_\mu \overline{\Psi}\gamma^\mu\Psi - \frac{1}{4}\delta_A F_{\mu\nu}F^{\mu\nu} + i\delta_\Psi \overline{\Psi}\not{\partial}\Psi - \delta_e A_\mu \overline{\Psi}\gamma^\mu\Psi$$

where $\delta_A = Z_A - 1$, $\delta_\Psi = Z_\Psi - 1$, and $\delta_e = e_0 Z_A^{\frac{1}{2}} Z_\Psi - e$.

Integrating by parts the photonic counterterm yields

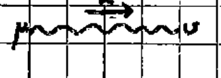
$$S_\gamma = \int d^4x \left(-\frac{1}{4}\delta_A F_{\mu\nu}F^{\mu\nu}\right) = -\frac{1}{4}\delta_A \int d^4x \big(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu - \partial_\nu A_\mu \partial^\mu A^\nu + \partial_\nu A_\mu \partial^\nu A^\mu\big)$$

$$= -\frac{1}{2}\delta_A \int d^4x \left(\partial_\mu A_\nu \partial^\mu A^\nu - \partial_\mu A_\nu \partial^\nu A^\mu\right)$$

$$= -\frac{1}{2}\delta_A \int d^4x \left(-A^\nu \partial_\mu \partial^\mu A^\nu + A_\nu \partial_\mu \partial^\nu A^\mu\right)$$

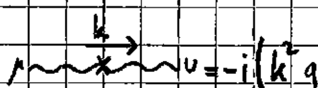
$$= -\frac{1}{2}\delta_A \int d^4x\, A_\nu \left(-\partial^2 g^{\mu\nu} + \partial^\mu\partial^\nu\right) A_\mu .$$

So our complete set of Feynman rules yields


$= \frac{-ig_{\mu\nu}}{k^2}$

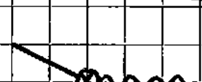

$= \frac{i}{\not{p}}$


$= -ie\gamma^\mu$

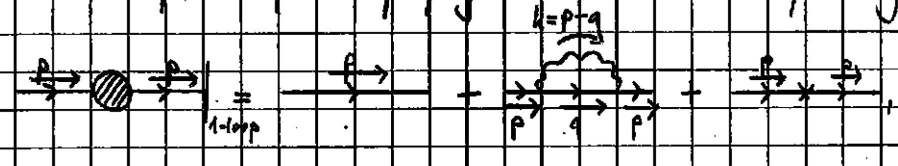

$= -i\left(k^2 g_{\mu\nu} - k_\mu k_\nu\right)\delta_A$


$= i\left(\not{p}\,\delta_\Psi - \delta_m\right)$

needs to be added even in massless theories

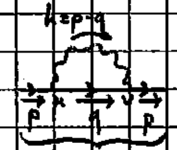

$= -ie\gamma^\mu$

At one-loop, the photon propagator receives the following contributions



of which only the latter two are corrections. We compute

$$= i(\not p\, \delta_\psi - \delta_m)$$



$$= \bar u(p)(-ie\gamma^\nu)\int \frac{d^d q}{(2\pi)^d}\frac{i\not q}{q^2}\frac{-ig_{\mu\nu}}{k^2}(-ie\gamma^\mu)u(p)$$

$$-i\mathcal{M}(f(p)\to f(p)) = -e^2\,\bar u(p)\int\frac{d^d q}{(2\pi)^d}\frac{\gamma_\mu \not q\,\gamma^\mu}{q^2(p-q)^2}\,u(p), \quad \gamma_\mu \not q \gamma^\mu = \gamma_\mu(-\gamma^\mu \not q + 2q^\mu) = -2\not q$$

$$= -e^2\,\bar u(p)\frac{\Gamma(1+1)}{\Gamma(1)\Gamma(1)}\int_0^1 dx\int\frac{d^d q}{(2\pi)^d}\frac{-2\not q}{(\underbrace{x(p-q)^2+(1-x)q^2}_{xp^2-2xpq+xq^2+q^2-xq^2})^2}$$

$$= 2e^2\,\bar u(p)\int_0^1 dx\int\frac{d^d q}{(2\pi)^d}\frac{\not q}{((\underbrace{q-xp}_{l})^2+x(1-x)p^2)^2}\,u(p)$$

$$= 8\pi\alpha_{em}\,\bar u(p)\int_0^1 dx\int\frac{d^d l}{(2\pi)^d}\frac{\overbrace{\not l}^{0,\text{ odd}}+x\not p}{(l^2+x(1-x)p^2)^2}\,u(p)$$

$$= 8\pi\alpha_{em}\,\bar u(p)\,\not p\,F(p^2)\,u(p), \qquad (\not p\, u(p) = 0)$$

where we calculate $F(p^2)$ using the subst. $z = \frac{\Delta}{l_E^2-\Delta}$, $l_E = \Delta^{\frac12}\left(\frac1z - 1\right)^{\frac12}$,

$$F(p^2) = i\int_0^1 dx\int\frac{d^d l_E}{(2\pi)^d}\frac{x}{(l_E^2 - \underbrace{x(1-x)p^2}_{\Delta})^2} = i\int_0^1 dx\,x\int\frac{d\Omega_d}{(2\pi)^d}\int_0^\infty dl_E\,\frac{l_E^{d-1}}{(l_E^2-\Delta)^2}$$

$$= i\int_0^1 dx\,x\,\frac{2\pi^{\frac d2}}{(2\pi)^d\Gamma(\frac d2)}\int_1^0 dz\,\frac{(l_E-\Delta)^2}{-2\Delta l_E}\frac{l_E^{d-1}}{(l_E-\Delta)^2}$$

$$= \frac{i}{(4\pi)^{\frac d2}\Gamma(\frac d2)}\int_0^1 dx\,x\int_0^1 dz\,\Delta^{\frac d2-2}z^{1-\frac d2}(1-z)^{\frac d2-1}$$

$$= \frac{i}{(4\pi)^{\frac d2}\Gamma(\frac d2)}\frac{\Gamma(2-\frac d2)\Gamma(\frac d2)}{\Gamma(2-\frac d2+\frac d2)}\int_0^1 dx\,x\,\Delta^{\frac d2-2} = \frac{i\Gamma(\frac\epsilon2)}{(4\pi)^{d/2}}\int_0^1 dx\,x\,(x(1-x)p^2)^{\frac d2-2}.$$

To bring out the divergence, we expand the integrand, 09.07.2015

$$\int_0^1 dx\, x\,(x(1-x)p^2)^{\frac{\epsilon}{2}} = \int_0^1 dx\, x\left(1+\frac{\epsilon}{2}\ln(x(1-x)p^2)+\sigma(\epsilon^2)\right)$$

$$=\frac{1}{2}+\frac{\epsilon}{2}\int_0^1 dx\, x\,\ln(x(1-x)p^2)+\sigma(\epsilon^2).$$

We can also expand the prefactors $\Gamma(\frac{\epsilon}{2})=\frac{2}{\epsilon}-\gamma+\sigma(\epsilon)$ and

$$\frac{1}{(4\pi)^{d/2}}=\frac{1}{(4\pi)^{2-\frac{\epsilon}{2}}}=\frac{(4\pi)^{\frac{\epsilon}{2}}}{16\pi^2}=\frac{1}{16\pi^2}\left(1+\frac{\epsilon}{2}\ln(4\pi)+\sigma(\epsilon^2)\right).$$

Assembling everything, we get

$$F(p^2)=\frac{i}{16\pi^2}\left(1+\frac{\epsilon}{2}\ln(4\pi)+\sigma(\epsilon^2)\right)\left(\frac{2}{\epsilon}-\gamma+\sigma(\epsilon)\right)\left(\frac{1}{2}+\frac{\epsilon}{2}\int_0^1 dx\, x\ln(x(1-x)p^2)+\sigma(\epsilon^2)\right)$$

$$=\frac{i}{32\pi^2}\left(\frac{2}{\epsilon}-\gamma+\ln(4\pi)+\int_0^1 dx\, x\ln(x(1-x)p^2)\right)+\sigma(\epsilon)$$

If we choose our renormalization conditions as

$$\Sigma(\not p)|_{\not p=\mu}=0 \qquad \text{and} \qquad \frac{d}{d\not p}\Sigma(\not p)|_{\not p=\mu}=0,$$

we obtain a counterterm δ_ψ of

$$\frac{d}{d\not p}\Sigma(\not p)|_{\not p=\mu}=i\delta_\psi+\frac{d}{d\not p}\left[2e^2\frac{i}{32\pi^2}\left(\frac{2}{\epsilon}-\gamma+\ln(4\pi)+\int_0^1 dx\, x\ln(x(1-x)\not p^2)\right)\right]\Big|_{\not p=\mu}$$

$$=i\delta_\psi+\frac{ie^2}{16\pi^2}\int_0^1 dx\,\frac{2x(1-x)\not p}{x(1-x)\not p^2}\Big|_{\not p=\mu}=i\delta_\psi+\frac{ie^2}{8\pi^2\mu}=0 \implies \delta_\psi=-\frac{e^2}{8\pi^2\mu}$$

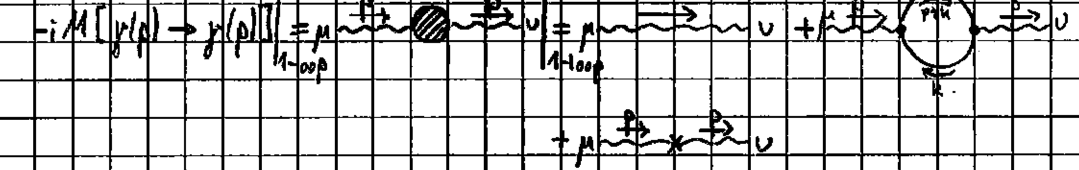
and for δ_m

$$\Sigma(\not p)|_{\not p=\mu}=i(\not p\,\delta_\psi-\delta_m)+\frac{ie^2}{16\pi^2}\left(\frac{2}{\epsilon}-\gamma+\ln(4\pi)+\int_0^1 dx\, x\ln(x(1-x)\not p^2)\right)\Big|_{\not p=\mu}$$

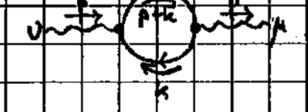
$$=-\frac{ie^2}{8\pi^2}-i\delta_m+\frac{ie^2}{16\pi^2}\left(\frac{2}{\epsilon}-\gamma+\ln(4\pi)+\int_0^1 dx\, x\ln(x(1-x)\mu^2)\right)=0$$

$$\implies \delta_m=\frac{e^2}{16\pi^2}\left(\frac{2}{\epsilon}-\gamma+\ln(4\pi)+\int_0^1 dx\, x\ln(x(1-x)\mu^2)-2\right).$$

2. We calculate the one-loop corrections to the photon propagator, again using the Feynman rules of renormalized perturbation theory.

$-i\mathcal{M}[\gamma(p)\to\gamma(p)]\big|_{1\text{-loop}} =$ 

The first correction gives

 $= \epsilon^*_\mu(p)\left(-ie(\gamma^\mu)^A{}_B\right)\int\frac{d^dp}{(2\pi)^d}\frac{i(\not p+\not k)^B{}_C}{(p+k)^2}\frac{i(\not k)^C{}_D}{k^2}\left(-ie(\gamma^\nu)^D{}_A\right)\epsilon_\nu(p)$

$$= -e^2\epsilon^*_\mu(p)\int\frac{d^dk}{(2\pi)^d}\frac{Tr(\gamma^\mu(\not p+\not k)\gamma^\nu\not k)}{k^2(p+k)^2}\epsilon_\nu(p)$$

$$Tr(\gamma^\mu(\not p+\not k)\gamma^\nu\not k) = p_\rho k_\sigma Tr(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma) + Tr(\gamma^\mu\not k k_\rho(2\eta^{\rho\nu}-\gamma^\rho\gamma^\nu))$$

$$= p_\rho k_\sigma 4(\eta^{\mu\rho}\eta^{\nu\sigma}-\eta^{\mu\nu}\eta^{\rho\sigma}+\eta^{\mu\sigma}\eta^{\rho\nu}) + 2Tr(\gamma^\mu\not k k^\nu) - k^2 Tr(\gamma^\mu\gamma^\nu)$$

$$= 4(p^\mu k^\nu - p^\sigma k_\sigma\eta^{\mu\nu} + p^\nu k^\mu) + 2k_\rho k^\nu Tr(\gamma^\mu\gamma^\rho) - k^2 4\eta^{\mu\nu}$$

$$= 4(p^\mu k^\nu - p\cdot k\,\eta^{\mu\nu} + p^\nu k^\mu) + 8k^\mu k^\nu - 4k^2\eta^{\mu\nu}$$

Using a Feynman parametrization to simplify the denominator

$$i\mathcal{M}[\gamma(p)\to\gamma(p)] = -e^2\epsilon^*_\mu(p)\int\frac{d^dk}{(2\pi)^d}\int_0^1 dx\,\frac{Tr(\gamma^\mu(\not p+\not k)\gamma^\nu\not k)}{[x(p+k)^2+(1-x)k^2]^2}\epsilon_\nu(p)$$

$$x(p+k)^2+(1-x)k^2 = xp^2+2xpk+xk^2+k^2-xk^2$$

$$= (k+xp)^2 + x(1-x)p^2 =: l^2 + x(1-x)p^2 = l^2-\Delta$$

where $l = k+xp$, $dl = dk$

$$i\mathcal{M} = -e^2\epsilon^*_\mu(p)\,Tr(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma)\int_0^1 dx\int\frac{d^dl}{(2\pi)^d}\frac{\overbrace{(l_\rho+(1-x)p_\rho)(l_\sigma-xp_\sigma)}^{l_\rho l_\sigma - x(1-x)p_\rho p_\sigma,\ \text{other terms odd}}}{(l^2-\Delta)^2}\epsilon_\nu(p)$$

$$= -e^2\epsilon^*_\mu(p)\,Tr(\gamma^\mu\gamma^\rho\gamma^\nu\gamma^\sigma)\int_0^1 dx\left[\frac{-i\Gamma(1-\frac{d}{2})}{(4\pi)^{d/2}}\frac{\eta_{\rho\sigma}}{2}\Delta^{\frac{d}{2}-1} - x(1-x)p_\rho p_\sigma\frac{i\Gamma(2-\frac{d}{2})}{(4\pi)^{d/2}}\Delta^{\frac{d}{2}-2}\right]\epsilon_\nu(p),$$

where we used A.44 and A.46 from Peskin & Schroeder to perform the L-integration. We simplify

$$iM = \frac{ie^2 \epsilon^*_\mu}{(4\pi)^{d/2}} \Gamma\left(2-\frac{d}{2}\right) Tr\left(\gamma^\mu \gamma^\rho \gamma^\nu \gamma^\sigma\right) \int_0^1 dx \left(\frac{\eta_{\rho\sigma}}{2-d} \left(-x(1-x)p^2\right)^{\frac{d}{2}-1} + x(1-x) p_\rho p_\sigma \left(-x(1-x)p^2\right)^{\frac{d}{2}-2} \right) \epsilon_\nu(p)$$

$$= \frac{ie^2}{(4\pi)^{d/2}} \epsilon^*_\mu(p) \Gamma\left(\frac{\epsilon}{2}\right) Tr\left(\gamma^\mu \gamma^\rho \gamma^\nu \gamma^\sigma\right) \left(\frac{-p^2 \eta_{\rho\sigma}}{2-d} + p_\rho p_\sigma \right) (-p^2)^{\frac{d}{2}-2} \int_0^1 dx \, 2\left[x(1-x)\right]^{\frac{d}{2}-1} \epsilon_\nu(p)$$

$$= \frac{ie^2}{(4\pi)^{d/2}} \epsilon^*_\mu(p) \Gamma\left(\frac{\epsilon}{2}\right) Tr\left(\gamma^\mu \gamma^\rho \gamma^\nu \gamma^\sigma\right) (-p^2)^{-\frac{\epsilon}{2}} \left(\frac{-p^2}{2-d} \eta_{\rho\sigma} + p_\rho p_\sigma \right) \frac{\Gamma\left(\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma(d)} \epsilon_\nu(p)$$

$$= \frac{ie^2}{(4\pi)^{d/2}} \epsilon^*_\mu(p) \frac{\Gamma\left(\frac{\epsilon}{2}\right)\Gamma\left(\frac{d}{2}\right)^2}{\Gamma(d)} (-p^2)^{-\frac{\epsilon}{2}} 4\left(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\nu}\eta^{\rho\sigma} + \eta^{\mu\sigma}\eta^{\rho\nu}\right) \left(\frac{-p^2}{2-d} \eta_{\rho\sigma} + p_\rho p_\sigma \right) \epsilon_\nu(p)$$

$$= \frac{4ie^2}{(4\pi)^{d/2}} \epsilon^*_\mu(p) \frac{\Gamma\left(\frac{\epsilon}{2}\right)\Gamma\left(\frac{d}{2}\right)^2}{\Gamma(d)} (-p^2)^{-\frac{\epsilon}{2}} \left(\frac{-p^2}{2-d} \left(\eta^{\mu\nu} - \eta^{\mu\nu} \underbrace{\eta^\rho_\rho}_{d} + \eta^{\mu\nu} \right) + \left(p^\mu p^\nu - \eta^{\mu\nu} p^2 + p^\mu p^\nu \right) \right) \epsilon_\nu(p)$$

$$= \frac{4ie^2}{(4\pi)^{d/2}} \epsilon^*_\mu(p) \frac{\Gamma\left(\frac{\epsilon}{2}\right)\Gamma\left(\frac{d}{2}\right)^2}{\Gamma(d)} (-p^2)^{-\frac{\epsilon}{2}} \left(-\frac{2\eta^{\mu\nu} p^2}{2-d} + \frac{d \eta^{\mu\nu} p^2}{2-d} + 2p^\mu p^\nu - \eta^{\mu\nu} p^2 \right) \epsilon_\nu(p)$$

$$= \frac{4ie^2}{(4\pi)^{d/2}} \epsilon^*_\mu(p) \frac{\Gamma\left(\frac{\epsilon}{2}\right)\Gamma\left(\frac{d}{2}\right)^2}{\Gamma(d)} (-p^2)^{-\frac{\epsilon}{2}} \left(-2\eta^{\mu\nu} p^2 + 2p^\mu p^\nu \right) \epsilon_\nu(p)$$

$$= \frac{8ie^2}{(4\pi)^{d/2}} \epsilon^*_\mu(p) \frac{\Gamma\left(\frac{\epsilon}{2}\right)\Gamma\left(\frac{d}{2}\right)^2}{\Gamma(d)} (-p^2)^{-\frac{\epsilon}{2}+1} \left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \epsilon_\nu(p)$$

$$= \epsilon^*_\mu(p) \, \alpha \frac{8i}{(4\pi)^{d/2-1}} \frac{\Gamma\left(\frac{\epsilon}{2}\right)\Gamma\left(\frac{d}{2}\right)^2}{\Gamma(d)} (-p^2)^{1-\frac{\epsilon}{2}} \left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2} \right) \epsilon_\nu(p)$$

$$= \epsilon^*_\mu(p) \, \Pi^{\mu\nu}(p^2) \, \epsilon_\nu(p)$$

We need $\Pi^{\mu\nu}(p^2) \propto \left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)$ to fulfill the Ward identity. Physically, this is just current conservation which arises as a consequence of the gauge symmetry of QED. With $\Pi^{\mu\nu}(p^2) \propto \left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right)$, we get

$$p_\mu \Pi^{\mu\nu}(p^2) \propto p_\mu \left(\eta^{\mu\nu} - \frac{p^\mu p^\nu}{p^2}\right) = p^\nu - \frac{p^2 p^\nu}{p^2} = 0.$$