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Exam on Quantum Field Theory I

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Problem 1 (12 points) - Quantization of a complex scalar field

A free complex-valued scalar field is described by the Lagrangian density

$$\mathcal{L} = \partial_\mu \phi^+ \partial^\mu \phi - m^2 \phi^+ \phi. \tag{1}$$

a) Show that the above Lagrangian is invariant under the global phase transformation $\phi \longrightarrow e^{i\alpha}\phi$. What do we call these transformations in the language of field theory? What are the physical implications of such invariances?

$$\phi \longrightarrow e^{i\alpha}\phi \implies \phi^+ \longrightarrow (e^{i\alpha}\phi)^+ = \phi^+ e^{-i\alpha}$$

$$\mathcal{L} \longrightarrow \mathcal{L}' = \partial_\mu \phi^+ e^{-i\alpha} \partial^\mu e^{i\alpha} \phi - m^2 \phi^+ e^{-i\alpha} e^{i\alpha} \phi = \partial_\mu \phi^+ \partial^\mu \phi - m^2 \phi^+ \phi = \mathcal{L}$$

Such a global phase transformation is classified as $U(1)$. Underlying the Lagrangian's invariance is a continuous symmetry. According to Noether's theorem such symmetries invoke conserved charges, i.e. physical quantities.

b) Find the conjugate momenta associated to each component.

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot\phi(x)} = \dot\phi^+(x), \qquad \pi^+(x) = \frac{\partial \mathcal{L}}{\partial \dot\phi^+(x)} = \dot\phi(x). \tag{2}$$

c) Show that the corresponding Hamiltonian H can be cast as

$$H = \int d^3x\, \mathcal{H}(x) = \int d^3x \left(\pi(x)\, \dot\phi(x) + \pi^+(x)\, \dot\phi^+(x) - \mathcal{L}(x) \right)$$

$$= \int d^3x \left(\pi(x)\, \pi^+(x) + \vec\nabla \phi^+(x)\, \vec\nabla \phi(x) + m^2 \phi^+(x)\, \phi(x) \right) \tag{3}$$

$$\pi(x)\dot\phi(x) + \pi^+(x)\dot\phi^+(x) - \mathcal{L}(x) = \dot\phi^+\dot\phi + \dot\phi\dot\phi^+ - \dot\phi^+\dot\phi + \vec\nabla\phi^+\vec\nabla\phi + m^2\phi^+\phi$$

$$= \pi^+\pi + \vec\nabla\phi^+\vec\nabla\phi + m^2\phi^+\phi$$

d) The corresponding quantized fields we can write as

$$\phi(x) = \int \tilde{dk} \left(a(k)\, e^{-ikx} + b^{+}(k)\, e^{ikx} \right), \quad \phi^{+}(x) = \int \tilde{dk} \left(a^{+}(k)\, e^{ikx} + b(k)\, e^{-ikx} \right).$$

With these expressions at hand, verify that the Hamiltonian (3) takes on the diagonal form

$$H = \int d^3x\, \mathcal{H}(x) = \int \tilde{dk}\; E_k \left(a^{+}(k)\, a(k) + b^{+}(k)\, b(k) \right)$$

$$\dot{\phi}(x) = -i \int \tilde{dk}\, E_k \left(a(k)\, e^{-ikx} - b^{+}(k)\, e^{ikx} \right), \quad \dot{\phi}^{+}(x) = i \int \tilde{dk}\, E_k \left(a^{+}(k)\, e^{ikx} - b(k)\, e^{-ikx} \right).$$

$$\vec{\nabla}\phi(x) = i \int \tilde{dk}\, \vec{k} \left(a(k)\, e^{-ikx} - b^{+}(k)\, e^{ikx} \right), \quad \vec{\nabla}\phi^{+}(x) = -i \int \tilde{dk}\, \vec{k} \left(a^{+}(k)\, e^{ikx} - b(k)\, e^{-ikx} \right).$$

$$H = \int d^3x\, \mathcal{H}(x) = \int d^3x \left(\dot{\phi}\, \dot{\phi}^{+} + \vec{\nabla}\phi^{+} \vec{\nabla}\phi + m^2 \phi^{+} \phi \right)$$

$$= \int d^3x \int \tilde{dk}\, E_k \left(a(k)\, e^{-ikx} - b^{+}(k)\, e^{ikx} \right) \int \tilde{dp}\, E_p \left(a^{+}(p)\, e^{ipx} - b(p)\, e^{-ipx} \right)$$

$$+ \int d^3x \int \tilde{dk}\, \vec{k} \left(a(k)\, e^{-ikx} - b^{+}(k)\, e^{ikx} \right) \int \tilde{dp}\, \vec{p} \left(a^{+}(p)\, e^{ipx} - b(p)\, e^{-ipx} \right)$$

$$+ m^2 \int d^3x \int \tilde{dk} \left(a(k)\, e^{-ikx} + b^{+}(k)\, e^{ikx} \right) \int \tilde{dp} \left(a^{+}(p)\, e^{ipx} + b(p)\, e^{-ipx} \right)$$

$$= \int \tilde{dk}\; E_k^2 \; a(k)\, a^{+}(k) - a(k)\, b(-k)\, e^{-2iE_k t} - b^{+}(k)\, a^{+}(-k)\, e^{2iE_k t} + b^{+}(k)\, b(k))$$

$$+ \int \tilde{dk}\; \vec{k}^2 \left(a(k)\, a^{+}(k) + a(k)\, b(-k)\, e^{-2iE_k t} + b^{+}(k)\, a^{+}(-k)\, e^{2iE_k t} + b^{+}(k)\, b(k) \right)$$

$$+ \int \tilde{dk}\; m^2 \left(a(k)\, a^{+}(k) + a(k)\, b(-k)\, e^{-2iE_k t} + b^{+}(k)\, a(-k)\, e^{2iE_k t} + b^{+}(k)\, b(k) \right)$$

$$= 2 \int \tilde{dk}\; E_k^2 \left(a(k)\, a^{+}(k) + b^{+}(k)\, b(k) \right) = 2 \int \tilde{dk}\; E_k^2 \left(a^{+}(k)\, a(k) + (2\pi)^3\, 2E_k\, \delta^{(3)}(0) + b^{+}(k)\, b(k) \right)$$

$$= 2 \int \tilde{dk}\; E_k^2 \left(a^{+}(k)\, a(k) + b^{+}(k)\, b(k) \right) + 4(2\pi)^3 \int \tilde{dk}\; E_k^3\, \delta^{(3)}(0)$$