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Group Theory - Lecture 6

6. Lie groups / Lie algebras

- Noether's theorem
- symmetries of space-time: Poincaré-symmetries
- gauge-symmetries: $SU(3) \times SU(2) \times U(1)$

Def: A Lie group is a group which is also a smooth manifold, such that

$$\left.\begin{array}{ll} m: G \times G \longrightarrow G & \text{(multiplication)} \\ (\cdot)^{-1}: G \longrightarrow G & \text{(inverse)} \end{array}\right\} \text{are smooth maps}$$

All examples we will consider are matrix groups, i.e. subgroups of $GL(n)$, as such they are submanifolds of $\mathbb{R}^{n^2}/\mathbb{C}^{n^2}$.
Often, they are described as level sets

$\{f = 0\} \subset GL(n)$

for some smooth $\underset{\substack{\uparrow \\ \text{derivative of } f \text{ has maximum rank everywhere}}}{\text{regular}}$ function $f: \mathbb{R}^n \to \mathbb{R}^k$

Examples

i) $(\mathbb{R}, +)$ manifold

$$\left.\begin{array}{l} m(x,y) = x + y \\ (x)^{-1} = -x \end{array}\right\} \text{smooth maps}$$

ii) $U(1) = \{z \in \mathbb{C} \mid |z|^2 = 1\} = \{e^{i\varphi} \mid \varphi \in \mathbb{R}\} \cong S^1$ (unit sphere)

$$\left.\begin{array}{l} m(z,w) = zw \\ (z)^{-1} = \bar{z} \end{array}\right\} \text{smooth maps}$$

iii) (special) unitary group

$(S)U(n) = \{A \in Mat(n,n,\mathbb{C}) \mid A^\dagger A = \mathbb{1}, (\det A = +1)\}, \quad A^\dagger = \bar{A}^t$

iv) (special) orthogonal group

$$(S)O(n) = \{A \in Mat(n,n,\mathbb{R}) \mid A^t A = \mathbb{1}, \ (\det A = +1)\}$$

v) infinite orthogonal groups

$$O(p,q) = \{A \in Mat(n,n,\mathbb{R}) \mid A^t \left(\begin{array}{c|c} \mathbb{1}_p & 0 \\ \hline 0 & -\mathbb{1}_q \end{array}\right) A = \left(\begin{array}{c|c} \mathbb{1}_p & 0 \\ \hline 0 & -\mathbb{1}_q \end{array}\right)\}, \quad n=p+q$$

leaves invariant signature of (p,q) bilinear form

e.g. Lorentz group $\mathcal{L} = O(1,3)$

vi) symplectic group

$$Symp(2n) = \{A \in Mat(2n,2n,\mathbb{R}) \mid A^t \left(\begin{array}{c|c} 0 & \mathbb{1}_n \\ \hline -\mathbb{1}_n & 0 \end{array}\right) A = \left(\begin{array}{c|c} 0 & \mathbb{1}_n \\ \hline -\mathbb{1}_n & 0 \end{array}\right)\}$$

vii) Poincaré group: symmetries of $\mathbb{R}^{1,3}$

$$P = \underbrace{O(1,3) \ltimes}_{} \mathbb{R}^{1,3}$$

semi-direct product $(A,a)(B,b) = (AB, a+Ab)$

Concepts in group theory carry over to Lie-groups (but require smoothness), e.g.

A homomorphism of a Lie group is a smooth group homomorph. $\phi: G \to H$.

A Lie-subgroup $H \subset G$ is a subgroup, s.t. the inclusion map is a smooth immersion.

Examples

i) $(\mathbb{R},+) \longrightarrow U(1), \quad x \longmapsto e^{ix}$ hom. of Lie groups

ii) $U(1) \longrightarrow U(1), \quad z \longmapsto z^n, \ n \in \mathbb{Z}$, hom. of Lie groups

iii) $\det \ GL(n) \longrightarrow (\mathbb{R}^+, \cdot)$ is a hom. of Lie groups

iv) $O(n) \subset O(n+m), \quad A \longmapsto \left(\begin{array}{c|c} A & 0 \\ \hline 0 & \mathbb{1}_m \end{array}\right)$ Lie subgroup

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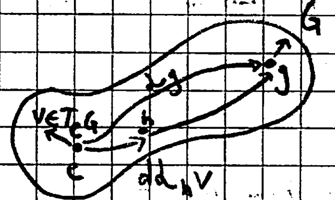
v) $U(1) \subset SU(2)$, $z \longmapsto \begin{pmatrix} z & 0 \\ 0 & \bar{z} \end{pmatrix}$ Lie subgroup

Combination of group theory with smooth manifolds
↳ remarkable simplification by linearization

Reason: All structures compatible with group-multiplications are determined by their properties around $e \in G$.

Left-multiplication map $L_g : G \longrightarrow G$, $h \longmapsto gh$ is a bijection of Lie groups.

$(dL_g)_e : T_e G \longrightarrow T_g G$



⟶ you get a vector field

$X_v(g)$: left-invariant vector field

$$\boxed{(dL_g)_h X_v(h) = X_v(gh)}$$

Much of the structure of G is captured by these vector fields.

$$\boxed{X_v(g) := (dL_g)_e v}$$

dL_g is the derivative of the map L_g, $L_e = id_G$
$dL_g = (L_g)_*$

Aside about vector fields:

In local coordinates

$$\boxed{X = \sum_i X_i \partial_i}$$ vector field on manifold M

$f : M \longrightarrow \mathbb{R}$

$$X(f) = \sum_i X_i (\partial_i f)$$

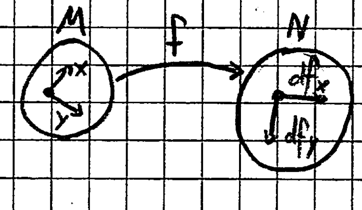
$$X(Yf) = \sum_{ij} X_i \partial_i (Y_j \partial_j f) = \sum_{ij} X_i Y_j (\partial_i \partial_j f) + (X_i \partial_i Y_j) \partial_j f$$

$$[X,Y](f) = X(Yf) - Y(Xf) = \sum_{i,j} (X_i \partial_i Y_j - Y_i \partial_i X_j)\partial_j f = Z(f)$$

$$Z_j = \sum_i (X_i \partial_i Y_j - Y_i \partial_i X_j)$$

$[\cdot,\cdot]: vf \times vf \longrightarrow vf$ Lie bracket of vector fields

Lie bracket: "calculates how a vector field changes along another one"



i) $f: M \longrightarrow N$

$$[dfX, dfY] = df([X,Y])$$

ii) Jacobi-identity

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

From i) it follows, that

left-invariant vector fields close under the Lie bracket

left-inv.: $dL_g X_v = X_v$

$$\underset{\displaystyle [X_v, X_w]}{\underset{\shortparallel}{[dL_g X_v, dL_g X_w]}} = dL_g([X_v, X_w])$$

$\Longrightarrow$ Lie bracket of left-inv. vector fields is left-inv.

$\longrightarrow$ T_eG is equipped with the structure of a Lie algebra

$$v, w \in T_eG: \ [v,w] = X_{[v,w]} = [X_v, X_w]$$

Def.: A Lie algebra is a vector space V together with a skew-symmetric, bilinear map

$$[\cdot,\cdot]: V \times V \longrightarrow V$$

satisfying the Jacobi-identity

(skew-symmetric: $[v,w] = -[w,v]$)

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Matrix Groups

$G \in GL(n)$, $\quad T_e GL(n) = Mat(n,n)$, $\quad GL(n) = \{A \in Mat(n,n) \mid \det A \neq 0\}$

$T_e G$: smooth paths $h(t) \in G$, $\quad h(0) = e$

$Mat(n,n) \ni v = \dot{h}(0) \in T_e G$

$\underbrace{\ell_g(h) = g h}_{\text{matrix multiplication}}$, $\quad d\ell_g v = \frac{d}{dt}\Big|_{t=0} \ell_g h(t) = \frac{d}{dt}\Big|_{t=0} g h(t)$

$= g \frac{d}{dt}\Big|_{t=0} h(t) = g \cdot v$

$X_v(g) = g \cdot v$

Lie bracket: $X = \sum_i x_i \partial_i$, $\quad X(f) = \sum_i x_i (\partial_i f)$, $\quad X(Yf) - Y(Xf) = [X,Y](f)$

$X_v f(g) = \partial_g f(g)\,(g \cdot v)$

$X_v X_w f(g) = \partial_g\left(\partial_g f(g)\,(g \cdot w)\right) g \cdot v = \partial_g^2 f(g)\,(g \cdot w, g \cdot v) + \partial_g f(g)\, g v w$

$[X_v, X_w]\, f(g) = \partial_g f(g)\, g\,(vw - wv) = \partial_g f(g)\, g\,([v,w])$

$[X_v, X_w](g) = g\,[v,w] = X_{[v,w]}$ matrix commutator

$T_e G \subset Mat(n,n)$ has the structure of Lie-algebra, where

$[v,w] = vw - wv$ matrix multiplication

(Matrix commutator satisfies the Jacobi-identity.)

Example: $G = O(n) = \{A \in Mat(n,n,\mathbb{R}) \mid A^t A = \mathbb{1}_n\}$

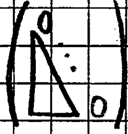
level set $G = \{f(A) = 0\}$, $\quad f(A) = A^t A - \mathbb{1}_n$

$T_e G = \ker(df_e)$

path $A(t)$ s.t. $A(0) = \mathbb{1}_n$, $\quad \dot{A}(0) = a$

$df_e\, a = \frac{d}{dt}\Big|_{t=0} f(A(t)) = \frac{d}{dt}\Big|_{t=0} \left(A^t A - \mathbb{1}_n\right) = a^t + a$

$T_eG = \{a \in Mat(n,n) \mid a^t = -a\}$



$\dim(G) = \frac{n}{2}(n-1)$

basis of T_eG $\quad E_{(i,j)} = \begin{array}{c c} & \begin{array}{cc} & j \\ & \downarrow \end{array} \\ \begin{array}{c} \\ j \rightarrow \end{array} & \begin{pmatrix} 0 \ddots & & 1 \\ & & \\ -1 & & 0 \end{pmatrix} \begin{array}{c} \leftarrow i \\ \\ \\ \end{array} \\ & \begin{array}{cc} \uparrow & \\ i & \end{array} \end{array} \quad i<j$

$[E_{(i,j)}, E_{(r,s)}] = \delta_{jr} E_{(i,s)} + \delta_{is} E_{(j,r)} - \delta_{ir} E_{(j,s)} - \delta_{js} E_{(i,r)}$

antisymmetric tensor

$n=3$: $\quad E_a = \varepsilon_{abc} E_{(b,c)}$

$[E_a, E_b] = \varepsilon_{abc} E_c$

Often, Lie algebras are given i.t.o. a basis.

<u>Def.:</u> A <u>homomorphism of Lie algebras</u> is a linear map

$\varphi: V \rightarrow W$, s.t.

$[\varphi(v), \varphi(w)] = \varphi([v,w])$.

<u>Def.:</u> A <u>sub-Lie algebra</u> $V' \subset V$ is a subspace s.t.

$[v,w] \in V' \quad \forall v,w \in V'$

Information about Lie-groups are contained in their Lie-algebras.

	Lie group		Lie algebra
i)	G	$\longmapsto$	$T_eG = \mathfrak{g}$
ii)	$\phi: G \rightarrow H$ hom. of Lie-group	$\longmapsto$	$\varphi := d\phi_e : \underbrace{T_eG}_{\mathfrak{g}} \rightarrow \underbrace{T_eH}_{\underline{h}}$ hom. of Lie-algebra
iii)	$H \subset G$ subgroup	$\longmapsto$	$T_eH = \underline{h} \supset \mathfrak{g} = T_eG$ subalgebra
iv)	unique simply connected Lie group $\hat{G}$ with Lie$(\hat{G}) = \mathfrak{g}$ *	$\longleftarrow$	$\mathfrak{g}$
v)	$\exists!$ <u>connected</u> $H \subset G$	$\longleftarrow$	$\underline{h} \subset \mathfrak{g}$

* but any $G = \hat{G}/\Gamma$, Γ discrete subgroup of $Z(\hat{G})$ also has the same Lie-alg.

ii) $\phi(gh) = \phi(g)\,\phi(h)$

$\phi \circ d_g = d_{\phi(g)} \circ \phi$

$d\phi \circ d d_g = d d_{\phi(g)} \circ d\phi$

$d\phi\, X_v(g) = X_{d\phi_e v}(\phi(g))$

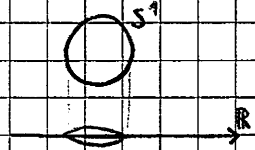
$$X_{[d\phi_e v,\, d\phi_e w]} = [X_{d\phi_e v}, X_{d\phi_e w}] = [d\phi\, X_v(g), d\phi\, X_w(g)]$$

$$= d\phi\, [X_v, X_w](g) = d\phi\, X_{[v,w]} = X_{d\phi_e [v,w]}$$

From this follows ii)

iii) This follows from looking at inclusion.

<u>Def.:</u> <u>Simply connected</u> means connected and there are no non-contractable loops, e.g. S^1 is <u>not</u> simply connected.



Continuing the table from previous page:

vi) $\exists!\ \phi: G \to H \quad \longleftrightarrow \quad \varphi: \underline{g} \to \underline{h}$

$d\phi_e = \varphi$

G simply connected, H connected

vii) $G \cong H \quad \longleftrightarrow \quad \underline{g} \cong \underline{h}$

if both are simply connected

In particular: repr. of $G \longmapsto$ repr. of $\underline{g}$

repr. of $G \longleftarrow$ repr. of $\underline{g}$ but only if G is simply conn.

If G is not simply connected, then not all repr. of $\underline{g}$ integrate to repr. of G.

E.g. $SO(3)$: $so(3) = \mathcal{L}ie(SO(3))$

$\quad\quad\quad\quad \cong$

$su(2) = \mathcal{L}ie(SU(2))$

Repr. of su(2) are denoted by spin $j \in \frac{1}{2}\mathbb{N}$, but only the integral spins give rise to repr. of SO(3).