5. REPRESENTATIONS of S_N

- Show up in quantum desc of many particle systems
- Play an important role in the representation theory of $SU(n)$.

$|S_n| = n' < \infty$ all complex repr. are fully decomp.

need to study irred. repres $\xleftrightarrow{1:1}$ conjugacy class

<u>Recall</u>

- conjugacy classes

$$S_n \to \pi = (i_1^1, \dots i_{r_1}^1)\cdot(i_1^2, \dots i_{r_2}^2) \cdot \dots \cdot (i_1^n \dots i_{r_n}^n)$$

$k_i :=$ # cycles of length i $\quad \sum i k_i = n$

$(k_1, .. k_2) \leftarrow$ cycle structure

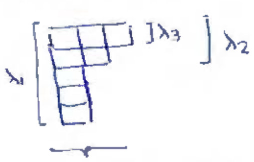
↳ determines the conjugacy class

- another parametrization by partitions: $\quad k_i = \lambda_i - \lambda_{i+1}$

$\lambda_i := $ # cycles of length $\geq i$ $\qquad \lambda_i = \sum_{j \geq i} k_j$

$\lambda_1 \geq \lambda_2 \geq \dots \qquad \sum \lambda_i = n$

pictorial repres. of partitions by means of Young diagrams



n boxes in colums of hight λ_i

$(k_1, k_2, \dots) \qquad (\lambda_1, \lambda_2 \dots$

$(2, 1, 1, 0 \dots) \sim (4, 2, 1, 0 \dots) \sim$ 

- <u>Regular representation</u>: on the group algebra $\mathbb{C}[G]$

$$V_{reg} = \mathbb{C}[G] = \{ \sum_{g \in G} \alpha_g \cdot g \mid \alpha_g \in \mathbb{C} \}$$

$$\left.\begin{aligned} &\lambda(\sum \alpha_g \cdot g) = \sum(\lambda \alpha_g)\cdot g \\ &\sum \alpha_g \cdot g + \sum \beta_g \cdot g = \sum(\alpha_g + \beta_g) g \\ &(\sum_{g \in G} \alpha_g\, g)\cdot(\sum_{h \in G} \beta_h \cdot h) = \sum_{g,h \in G} (\alpha_g \cdot \beta_h)(g \cdot h) = \\ &\qquad = \sum_{g \in G} \Big(\sum_{h \in G} (\alpha_{gh^{-1}} \beta_h) \Big) g \end{aligned}\right\} \text{Group algebra structure}$$

$$\rho_{reg}(g)\Big(\sum_h{}' \alpha_h h\Big) = \sum \alpha_h (gh) \overset{g\to}{\underset{\downarrow}{=}} \sum \alpha_{g^{-1}h}\, h$$

representation

$$\rho_{reg} = \rho_1^{\oplus d_1} \oplus \rho_2^{\oplus d_2} \oplus \dots \oplus \rho_k^{\oplus d_k} \qquad d_i = \dim(\rho_i)$$

→ irreducibles in V_{reg} are ideals in group algebra.

<u>Aim</u>: Construct irr. reprs. of S_n as subrepr. of ρ_{reg}!

① Start with a Young diagram:

$$\begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ & \ \\ \cline{1-2} \ \\ \cline{1-1} \ \\ \cline{1-1}\end{array} \longmapsto \begin{array}{|c|c|c|}\hline 1 & 5 & 7 \\ \hline 2 & 6 \\ \cline{1-2} 3 \\ \cline{1-1} 4 \\ \cline{1-1}\end{array} = T \quad \text{(Young tableau)}$$

diagram ⟼ Tableau

by filling in the numbers $1, \dots, n$, in such a way that numbers increase from left to right and top to bottom.

② $R(T) = \{\sigma \in S_n \mid \sigma \text{ leaves rows } (T) \text{ invariant}\}$

$C(T) = \{\sigma \in S_n \mid \sigma \text{ leaves colums } (T) \text{ invariant}\}$

groups of row- and colums- transf. of T.

$R(T) \cap C(T) = \{e\}$

<u>Example</u>: $T = \begin{array}{|c|c|}\hline 1 & 4 \\ \hline 2 \\ \cline{1-1} 3 \\ \cline{1-1}\end{array}$ $\quad R(T) = \{e, (14)\}$

$C(T) = \{e, (12), (23), (13)(123), (132)\}$

③ $a_T := \sum_{\sigma\in R(T)} \sigma \in \mathbb{C}(S_n)$; $b_T := \sum_{\sigma \in C(T)} \text{sgn}(\sigma) \in \mathbb{C}(S_n)$

↳ symmetrizes the row transf ↳ antisymm. the colum transf.

Young symmetrizer: $c_T := a_T \cdot b_T$

<u>Example</u>: i) $T = \begin{array}{|c|c|c|}\hline 1 & 2 & 3 \\ \hline\end{array}$ $\quad c_T = a_T = \sum_{\sigma\in S_3} \sigma = 3!\,\Pi_0$

← projector on trivial repr.

ii) $T = \begin{array}{|c|}\hline 1 \\ \hline 2 \\ \hline 3 \\ \hline\end{array}$ $\quad c_T = b_T = \sum_{\sigma \in S_3} \text{sgn}(\sigma)\cdot\sigma = 3!\,\Pi_1$

← projector on sign repr.

iii) $T = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array}$ $\quad c_T = \underbrace{(e+(13))}_{a_T}\cdot(e-(12)) = e+(13)-(12)-(132) = 3\Pi_2$

④ $V_T := \overset{A}{\overbrace{\mathbb{C}[S_n]}} \cdot c_T$ is a subrepres. of V_{reg}

$= \{a\, c_T \mid a \in A\}$

Examples

i) $\boxed{1|2|3} = T$

$C(T) = \{e\}$

$R(T) = \{e, (1\,2), (2\,3), (1\,3), (1\,2\,3), (1\,3\,2)\} \equiv S_3$

$\mathbb{C}(S_3)\cdot c_T = \mathbb{C}(S_3)\cdot \sum_{\sigma\in S_3} \sigma = \mathbb{C}\cdot c_T \leftarrow$ 1-dim

$$\left[\pi \sum_{\sigma\in S_3} \sigma = \sum_{\sigma\in S_3} (\pi\sigma) = \sum_{\sigma\in S_3} \sigma\right]$$

$$\rho_{reg}(\pi) c_T = \rho_{reg}(\pi) \sum_{\sigma\in S_3} \sigma = \sum_{\sigma\in S_3} (\pi\sigma) = \sum_{\sigma\in S_3} \sigma = c_T$$

$\rightarrow$ trivial repr. !

ii) $T = \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$ $R(T) = e$, $C(T) = S_3$

$$V_T = \mathbb{C}(S_3)\, c_T = \mathbb{C}(S_3) \sum_{\sigma\in S_3} \text{sign}(\sigma)\cdot\sigma = \mathbb{C} c_T$$

$$\rho_{reg}(\pi)\, c_T = \pi \sum_{\sigma\in S_3} \text{sign}(\sigma)\, \sigma =$$

$$= \sum_{\sigma\in S_3} \text{sign}(\sigma)\cdot(\pi\sigma) =$$

$$= \text{sign}(\pi)\cdot \sum_{\sigma\in S_3} \text{sign}(\pi\sigma)(\pi\sigma) =$$

$$= \text{sign}(\pi)\cdot \sum_{\sigma\in S_3} \text{sign}(\pi)\cdot\sigma =$$

$$= \text{sign}(\pi)\cdot c_T$$

$\hookrightarrow$ $\rho_{reg}|_{V_T}$ is the sign repres.

iii) $T = \begin{array}{|c|c|}\hline 1 & 3\\\hline 2 \\\cline{1-1}\end{array}$ $c_T = e + (1\,3) - (1\,2) - (1\,2\,3)$

$V_T = \mathbb{C}[S_3]\cdot c_T$ is 2 dimensional, spanned by c_T and $(12)\, c_T$

$(1\,3)\, c_T = c_T$

$\rho_{reg}|_{V_T}$ is the irreducible 2d repres.

Fact: (i) $\rho_{reg}|_{V_T}$ is irreducible subrepresentation of ρ_{reg}.

(ii) $V_T \cong V_{T'}$ iff $\lambda(T) = \lambda(T')$

→ irreducible repres. only depends on underlying Young diagram.

→ there are as many copies of an irreducible representation corresponding to $(\lambda_1, \dots \lambda_n)$ as there are Young tableau built on this Young diagram.

Recall.

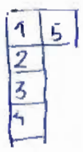
$$\rho_{reg} = \rho_1^{\oplus \dim(\rho_1)} \oplus \dots \oplus \rho_n^{\oplus \dim(\rho_n)}$$

→ irreducible represent. corresponding to a Young diagram has dimension the ~~#~~ number of ways you can make a Young tableau out of it.

Young diagram:

 ↦ irreducible repres. of S_n

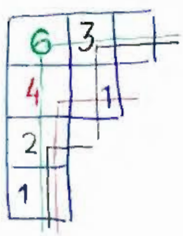
Young Tableau:



↦ subrepresentation of type  in ρ_{reg}.

- Hook-rule:

Formula for dimensions of irreducible representation of S_n.



Labels irreducible repres.

Hook numbers. draw a hook through a box and count the number of boxes met by the hook. h_i

$$\boxed{\dim(\rho_{\text{⊞}}) = \frac{n!}{\prod h_i}}$$ Hook rule

Examples:

i)  $\dim(\rho_{\square\square\cdots\square}) = 1$ trivial repres.

n-boxes

S_n $\quad d = \dfrac{n!}{n(n-1)\dots 1} = 1$

ii)  $\dim(\rho_{\boxminus}) = 1$ sign-repres

$d = \dfrac{n!}{n(n-1)\dots} = 1$

iii) $n-1$  $d = \dfrac{n!}{n(n-2)\dots 1} = (n-1)$

Same for 

- Frobenius formula (for characters of S_n)
  - denote irreduc repres by partitions $(\lambda_1, \dots \lambda_n)$
  - denote conjugacy classes by the cycle structure $(k_1, \dots k_n)$

  Define polynomials:

$$P_{(k_1 \dots k_n)}(x_1 \dots x_n) = \prod_{i<j}(x_i - x_j)(x_1 + \dots + x_n)^{k_1} \cdot (x_1^2 + \dots + x_n^2)^{k_2} \cdot (x_1^3 + \dots + x_n^3)^{k_3} \cdot \dots$$
$$\dots (x_1^n + \dots x_n^n)^{k_n}$$

$$\underset{(\lambda_1, \dots \lambda_n)}{\chi_\lambda}(\underset{(k_1 \dots k_n)}{C_k}) = \left[P_{(k_1 \dots k_n)}(x_1 \dots x_n)\right]_{x_1^{\ell_1} \dots x_n^{\ell_n}}$$

coeff. of in

$$\ell_i = \lambda_i + n - i$$

Example

$G = S_3$ $\quad P_{(1,1,0)}(x_1, x_2, x_3) = (x_1 - x_2)(x_1 - x_3)(x_2 - x_3)(x_1 + x_2 + x_3)(x_1^2 + x_2^2 + x_3^2)$

$(\cdot\cdot)(\cdot)$

 $\lambda = (3,0,0)$ $\quad \ell = (5,1,0)$ $\rightsquigarrow$ coeff. of $x_1^5 x_2$ in P $\quad \chi_\lambda(C_k) = 1$

 $\lambda = (1,1,1)$ $\quad \ell = (3,2,1)$ $\rightsquigarrow$ coeff. of $x_1^3 x_2^2 \cdot x_3$ $\quad \chi_\lambda(C_k) = -1$

 $\lambda = (2,1,0)$ $\quad \ell = (4,2,0)$ $\rightsquigarrow$ coef. of $x_1^4 x_2^2$ $\quad \chi_\lambda(C_k) = 0$

$\rightsquigarrow$ C_2-colum of characters table of $D_3 \cong S_3$.