2. FINITE GROUPS

A group G is <u>finite</u> if $|G| < \infty$. $|G|$ is called the <u>order</u> of G

Remark: (i) $H \subset G$ subgroup $\Rightarrow |H| < \infty$

$$G = \bigcup_{g \in G/H} gH \quad , \text{ multiplication is bijection } \Rightarrow |gH| = |H|$$

$$\Rightarrow |G| = |G/H| \cdot |H|$$

$|G/H| = [G : H]$ is sometimes called the index of H in G.

Orders of subgroups divide the order of the group!

[Lagrange's theorem]

(ii) $g \in G : \Gamma_g := \{ g^i \mid i \in \mathbb{Z} \} \subset G$

$|\Gamma_g| < \infty$

$\Rightarrow \exists n, m : g^n = g^m \Rightarrow \exists k \in \mathbb{N} : g^k = e$

$\Rightarrow \Gamma_g = \{ e, g, g^2, \dots, g^k = e \} \cong \mathbb{Z}_k$

$k =: \operatorname{ord}(g) = \operatorname{ord}(\Gamma_g)$

Lagrange theorem $\Rightarrow \operatorname{ord}(g) \mid |G|$

What does this mean if $|G|$ is prime?

$\Rightarrow \operatorname{ord}(g) = |G| \quad \forall e \neq g \in G$

$\Rightarrow \boxed{G \cong \mathbb{Z}_{|G|}}$

Examples:

(i) $\mathbb{Z}_{pq} \supset \mathbb{Z}_p$ generated by $[p]$

$\supset \mathbb{Z}_q$ generated by $[q]$

(ii) $|D_n| = 2n$

$$\left.\begin{array}{l} \Gamma_{S_i} = \{ e = R_0, S_i \} \\ \mathbb{Z}_n = \{ R_0, \dots, R_{n-1} \} \\ \quad \shortparallel \\ \Gamma_{R_1} \end{array}\right. \subset D_n$$

$\Gamma_{R_2} \cong \Gamma_{R_1}$, n odd $\qquad n = pq$, $\Gamma_{R_p} \cong \mathbb{Z}_q$

Finite group $G = \{g_1, \dots g_n\}$

Group is specified by <u>multiplication table</u>: — you can describe all groups by giving the multiplication tables.

G	g_1	g_2	$\cdots$	g_i	$\cdots$	g_n
g_1	g_1g_1	g_1g_2	$\cdots$	g_1g_i	$\cdots$	g_1g_n
g_2	g_2g_1		$\ddots$			
$\vdots$	$\vdots$					
g_j	g_jg_1			g_jg_i		
$\vdots$	$\vdots$					
g_n	g_ng_1					

Example:

D_3	R_0	R_1	R_2	S_0	S_1	S_2
R_0	R_0	R_1	R_2	S_0	S_1	S_2
R_1	R_1	R_2	R_0	S_1	S_2	S_0
R_2	R_2	R_0	R_1	S_2	S_0	S_1
S_0	S_0	S_2	S_1	R_0	R_2	R_1
S_1	S_1	S_0	S_2	R_1	R_0	R_2
S_2	S_2	S_1	S_0	R_2	R_1	R_0

Note: since multiplication is a bijection in G $\Rightarrow$ all rows (colums) are permutations of first row (colum).

$$g: \; g_i \longmapsto g g_i = g_{\pi_g(i)} \qquad \pi_g \in S_n$$

$$h(gg_i) = h(g_{\pi_g(i)}) = g_{\pi_h \circ \pi_g(i)}$$
$$\shortparallel$$
$$(hg)g_i = g_{\pi_{hg}(i)}$$

$\Rightarrow$ $\pi : g \mapsto \pi_g$ is group homomorphism

$$G \longrightarrow S_n$$

$\pi_g = e \in S_n \Rightarrow \pi_g(i) = i \;\; \forall i \Rightarrow g = e$

$\rightarrow \pi \cdot G \longleftrightarrow S_n$ is injective

$\Rightarrow G \cong \pi(G) \subset S_n$ all groups of order n are subgroups of S_n!

Importance of S_n for physics:

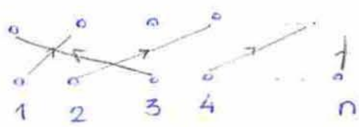
- statistics of particles in quantum systems.
- represent. of S_n govern the representation theory of $SU(N)$.

∘ Symmetric Group:

$S_n := B_{ij}\,(S = \{1, \dots, n\})$

$|S_n| = n!$

Notation:



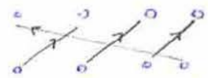
⇜ draw a row towards where the number goes

or

$$\pi = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \pi(1) & \pi(2) & \pi(3) & \cdots & \pi(n) \end{pmatrix}$$

Example: $S_4 \ni \pi$ cyclic permutation of $(1, 2, 3, 4)$

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix} =$$



Example: What is a Cayley subgroup of $\mathbb{Z}_4$?

$g_i = [i]$, $g_0 = e$, $g_1 = [1]$, $g_2 = [2]$, $g_3 = [3]$

$\pi : \mathbb{Z}_4 \longrightarrow S_4$

$g_i g_j = g_{i+j \bmod 4}$

$$\pi_e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}, \quad \pi_{[1]} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$$

$$\pi_{[2]} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}, \quad \pi_{[3]} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$$

$$H = \{\pi_e, \pi_{[1]}, \pi_{[2]}, \pi_{[3]}\} \subset S_4$$
$$\cong \mathbb{Z}_4$$

- Cycles:

special $\pi \in S_n$ which cyclically permute subsets of $S = \{1 \dots n\}$

$S_k \ni (s_0, \dots, s_{k-1})$: $s_i \longrightarrow s_{i+1 \bmod k}$

$r \in S_n \setminus \{s_1 \dots s_n) \mapsto r$

$(1\ 3\ 4) \in S_4$: $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 4 & 1 \end{pmatrix}$

There is a k-fold ambiguity in notation: $(s_0, s_1, \dots, s_k) = (s_1, s_2, \dots s_k, s_0)$

Non intersecting cycles commute:

$(s_1 \dots s_k)(s'_1 \dots s'_{k'}) = (s'_1 \dots s'_{k'})(s_1 \dots s_k)$

s.t. $\{s_1 \dots s_k\} \cap \{s'_1 \dots s'_{k'}\} = \phi$

Important fact:

All $\pi \in S_n$ can be written as product of non-intersecting cycles!

$\ulcorner$ Choose any $\pi \in S_n$

Define equivalent relation on $S = \{1, \dots n\}$

$x \sim y \iff \exists \ell \in \mathbb{N}_0 : y = \pi^{\ell}(x)$

$\Gamma_\pi \cong \mathbb{Z}_k$ for some k

$S = \dot{\bigcup}$ equivalence classes
$\quad\quad\quad\quad$ "
$\quad$ orbits under Γ_π

$S = S^0 \dot{\cup} S^1 \dot{\cup} \dots \dot{\cup} S^m$

$S^i = \{s_i, \pi(s_i), \pi^2(s_i), \dots, \pi^{n_i - 1}(s_i)\}$, $n_i = |S_i|$

$\pi = (s_1, \pi(s_1) \dots \pi^{n_1-1}(s_1))(s_2, \pi(s_2) \dots)(s_3, \dots) \dots (s_m, \pi(s_m), \dots \pi^{n_m-1}(s_m))$

product of non-intersecting cycles

Example:

$T = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 1 & 6 & 2 & 5 & 3 & 7 \end{pmatrix}$

$S = \{1, 4, 2\} \dot{\cup} \{3, 6\} \cup \{3\} \cup \{7\}$

$\pi = (1, 4, 2)(3, 6)$

Fact: under conjugation: cycles of particular length go to cycles of the same length.

$\sigma \in S_n \quad (s_1, \dots, s_k) \in S_n$

$\sigma(s_1, \dots, s_k)\sigma^{-1} = (\sigma(s_1), \dots, \sigma(s_k))$

Γ $\sigma(s_1 \dots s_k)\sigma^{-1}(\sigma(s_i)) = \sigma(s_1, \dots s_k)(s_i) = \sigma(s_{i+1})$

$\Rightarrow \sigma(s_1, \dots s_k)\sigma^{-1} = (\sigma(s_1) \dots \sigma(s_k))$ $\perp$

Conjugacy class is determined by the length-structure of the cyclic ~~group~~ decomp.

Conjugacy classes are characterized by cycle structure: $(k_1, \dots, k_n)$, where $k_i \geq 0$ is number of cycles of length i.

$\sum i k_i = n$

$$|C_{(k_1, \dots k_n)}| = \frac{n!}{\prod_i \underline{\underline{i}}^{k_i} \, k_i!}$$

all permutations

cyclic permutations in cycles

$(s_1 s_2 s_3) = (s_2 s_3 s_1)$

length of the cycle

n° of cycles with length i

permutations of cycles of same length

Reparametrisation: $(k_1 \dots k_n) \longmapsto (\lambda_1, \dots \lambda_n)$

λ_i is the number of cycles of length at least i

$$\lambda_i = \sum_{j=i}^{n} k_j$$

$k_i = \lambda_i - \lambda_{i+1}$

$$\sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{n} \sum_{j=1}^{n} k_j = \sum_{j} i k_j = n$$

$(\lambda_1, \dots \lambda_n)$ is a partition of n.

Characterize conjugacy classes

Cycle ~~struc~~ structures $\rightarrow \{(k_1, \dots, k_n) \mid k_i \geq 0, \sum i k_i = n\}$

$\updownarrow$ 1:1

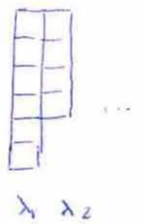
partitions of $n \rightarrow \{(\lambda_1, \dots \lambda_n\} \mid \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0, \sum \lambda_i = n\}$

nice way to visualise partitions by means of Young diagrams

$\updownarrow$

conjugacy classes of S_n

Young diagram associated to $(\lambda_1, \dots, \lambda_k)$ is a picture of n boxes in Colums of hights λ_i:



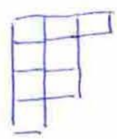
$\square \rightsquigarrow$ n boxes all together

Example: S_8: $(\cdot\cdot\cdot\cdot)(\cdot\cdot)(\cdot\cdot)(\cdot)$

$k_3 = 1$ $\quad \lambda_1 = 4$

$k_2 = 2$ $\quad \lambda_2 = 3$

$k_1 = 1$ $\quad \lambda_3 = 1$



conjugacy classes of S_n ?

$\quad\quad\quad \shortparallel$

# of partitions of n $= p(n)$

$p(1) = 1$

$p(2) = 2 \qquad 2 = 2+0,\ 1+1$

$p(3) = 3 \qquad 3 = 3+0,\ 2+1,\ 1+1+1$

$p(4) = 5$

$p(5) = 7$

$\vdots$

Generating functions:

$$\prod_{k\geq 1}\left(\frac{1}{1-x^k}\right) = \prod_k \sum_{i\geq 0} x^{ki} = \sum_n x^n p(n)$$

$$(1+x+x^2+\dots)(1+x^2+x^4+x^6+\dots)\cdot$$
$$\cdot(1+x^3+x^6+x^9+\dots)\ \dots$$

- Specific cycles:

 - transpositions:

$\sigma_i = (i, i+1)$: $\quad i \longmapsto i+1$
$\quad\quad i+1 \longmapsto i$
$\quad\quad r \longmapsto r,\ r \neq i+1$

$\sigma_i \sigma_i = e$

S_n is guaranteed by transpositions (all $\pi \in S_n$ can be ~~written~~ writen by products of transpositions)

$\ulcorner$ Take $\pi \in S_n$, $\pi(n) = i$

$\tilde{\pi} = \sigma_{n-1}\sigma_{n-2}\dots\sigma_i \pi$

$\tilde{\pi}(n) = n$

If $\tilde{\pi}$ is product of transpositions, then S_0 is $\pi = \sigma_i \sigma_{i+1} \dots \sigma_{k-1} \tilde{\pi}$

$S_2 = \{e, (12)\}$ is guaranted by transp.

$\text{length}(\pi) = \min\{k \mid \pi = \sigma_{i1} \dots \sigma_{ik}\}$

Define: $\varphi: S_n \longrightarrow \mathbb{Z}_2$

$$p(x_1 \dots x_n) = \prod_{i<j} (x_i - x_j)$$

$$\varphi(\pi) = \frac{p(x_{\pi(1)} \dots x_{\pi(n)})}{p(x_1, \dots x_k)} \in \{\pm 1\}$$

$$\varphi(\pi\sigma) = \underbrace{\frac{p(x_{\pi\sigma(1)} \dots x_{\pi\sigma(k)})}{p(x_1 \dots x_n)}}_{\varphi(\pi)} \times \underbrace{\frac{p(x_{\sigma(1)} \dots x_{\sigma(k)})}{p(x_{\sigma(1)} \dots x_{\sigma(n)})}}_{\varphi(\sigma)}$$

$\varphi(\sigma_i) = -1$

$\varphi(\pi) = (-1)^{\text{length}(\pi)}$

$\text{Ker}(\varphi) = \{\pi \mid \varphi(\pi) = 1\}$

$\overset{!!}{} \quad = \{\pi \text{ of even length}\} \trianglelefteq S_n$

A_n alternating group $\qquad S_n / A_n \cong \mathbb{Z}_2$

<u>Remark</u>: Finite group (simple) have been classified

- cyclic group $\mathbb{Z}_4$
- alternating group
- simple group of Lie-type
- one of 26 sporadic groups (Monster)