1. ABSTRACT GROUPS

Definition: A group G

$$G \times G \longrightarrow G$$
$$(g_1, g_2) \longmapsto g_1 g_2$$

s.t. (i) Associativity: $(g_1 g_2) g_3 = g_1 (g_2 g_3)$

(ii) Identity: $\exists e \in G : e \cdot g = g \ \forall g \in G$

(iii) Inverse: $\forall g \in G \ \exists g^{-1} : g^{-1} g = e$

In case: $\forall g, h \in G \ gh = hg$, then G is Abelian (commutative)

Remark: (i) right-inverses

$$g^{-1} g = e$$
$$\underset{\underset{g^{-1}(g g^{-1})}{\parallel}}{(g^{-1} g) g^{-1}} = e g^{-1} = g^{-1} \qquad \Rightarrow \boxed{g g^{-1} = e}$$

(ii) right-identity:

$$g \cdot e = g(g^{-1} g) = (g g^{-1}) g = eg = g$$

(iii) Uniqueness of e:

Assume $\exists e' : e' g = g \ \forall g$

$$e' \cdot e' = e = e'$$

(iv) uniqueness of inverse:

Assume that for $g \in G \ \exists h : hg = e$

$$h = h \cdot e = h(g g^{-1}) = (hg) g^{-1} = e g^{-1} = g^{-1}$$

Examples:

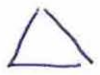
(i) $(\mathbb{Z}, +)$ Abelian group

(ii) Cycling groups $\mathbb{Z}_n = \mathbb{Z} / _n\mathbb{Z}$

(iii) Symmetric group S_n
(permutations of n objects)

(iv) Dihedral group D_3
(symmetries of a regular triangle Δ)

Example: Dihedral group D_3:

- D_3  Elements (6):

$$|D_3| = 6 = \{ e_0 = R_0,\ R_1 = R(\tfrac{2\pi}{3}),\ R_2 = R(\tfrac{4\pi}{3}),\ \underbrace{S_0, S_1, \cancel{R}S_2}_{\text{Reflections}} \}$$

$$\boxed{\begin{array}{l} R_i R_j = R_{i+j \bmod 3} \\ S_i^2 = R_0 \\ R_i S_j = S_{i+j \bmod 3} \end{array}}$$

↙ these properties define the group

Using this relations: ↙ these are consequences

$S_i R_j = S_{i-j}$

$S_i S_j = R_{-i+j}$ (check)

- D_n : sym. of regular n-gon

$$\begin{array}{l} \shortparallel \\ \{R_0, R_1, \dots, R_{n-1}, S_0, \dots, S_{n-1}\} \end{array} \qquad (*)$$

So far, all examples are discrete.

But now: Continuous ones.

Lie groups.

(v) translation group $(\mathbb{R}^n, +)$ Abelian.

(vi) General linear group.

$$GL(n) := \{ A \in Mat(n,n,\mathbb{R}) \mid \det(A) \neq 0 \}$$

(vii) (special) orthogonal group:

$$(S)\,O(n) = \{ A \in Mat(n,n;\mathbb{R}) \mid A^t A = \mathbb{1}_n \} \quad (\det(A) = 1)$$

(viii) Euclidean group (Sym. of a affine group)

$$E(n) := \{ (a, A) \in \mathbb{R}^n \times O(n) \}$$

$$(a, A)(a', A') = (a + Aa', AA')$$

(ix) (special) unitary group

$$(S)\,U(n) := \{ A \in Mat(n,n;\mathbb{R}) \mid \bar{A}^t = A^t A = \mathbb{1}_n \} \quad (\det(A) = 1)$$

Def: A group-homomorphism:

$\varphi : G \longrightarrow H$ s.t.

$$\varphi(g_1 g_2) = \varphi(g_1)\varphi(g_2)$$

- If φ is invertible, group isomorphism
- $\mathrm{Hom}(G,H)$ set of homomorphisms.

Rem: (i) $\varphi(e_G) = e_H$

$$\left[\varphi(g) = \varphi(g \cdot e_G) = \varphi(g)\cdot\varphi(e_G) \Rightarrow \varphi(e_G = e_H)\right]$$

(ii) $\varphi(g^{-1}) = (\varphi(g))^{-1}$

$$\left[\varphi(g^{-1})\varphi(g) = \varphi(g^{-1}g) = \varphi(e_G) = e_H \Rightarrow \varphi(g^{-1}) = (\varphi(g))^{-1}\right]$$

Examples:

(i) $\mathbb{Z}_n \longrightarrow D_n$

$[a] \longmapsto R_{a \bmod n}$ $\qquad R_i \cdot R_j = R_{i+j}$

(ii) $\mathbb{Z} \longrightarrow U(1) = \{a \in \mathbb{C} \mid \bar{a}a = 1\}$

$[a] \longmapsto e^{\frac{2\pi i a}{n}}$

(iii) $\varphi : O(2) \longrightarrow O(3)$

$A \longmapsto \left(\begin{array}{c|c} A & \begin{matrix} 0 \\ 0 \end{matrix} \\ \hline 0\,0 & 1 \end{array}\right)$

(iv) $U(n) \longrightarrow U(1)$

$A \longmapsto \det(A)$

Def: A subgroup $H \subset G$, which is itself a group w.r.t. multiplication in G

Rem. (i) H has to contain $e_G = e_H \overset{!}{\ni} H$

(ii) $H \subset G \Leftrightarrow \forall\, h_1, h_2 \in H \quad h_1 h_2^{-1} \in H$

(iii) Trivial subgroups $\{e\} \subset G$
$G \subset G$

(iv) G Abelian $\Rightarrow H \subset G$ also Abelian.

Examples

(i) $\mathbb{Z}_n \subset D_n$, $\mathbb{Z}_n = \{R_0, R_1, \dots, R_{n-1}\}$

(ii) $\underset{\substack{\cong \\ \{[0],\,[m][2m),\,\dots\,[(n-1)m]\}.}}{\mathbb{Z}_n} \subset \mathbb{Z}_{n\cdot m}$

(iii) $\underset{\substack{\cong \\ \{e^{i\varphi}\mathbb{1}_n \mid \varphi \in \mathbb{R}\}}}{U(1)} \subset U(n)$

(iv) $SO(n) \subset O(n)$

$SU(n) \subset U(n)$

(v) $O(n-1) \subset O(n)$

$$\in \begin{pmatrix} A & \\ & 1 \end{pmatrix}$$

(vi) $\forall g \in G : H = \Gamma_g = \{g^i \mid i \in \mathbb{Z}\}$

Abelian

(vii) center $Z(G) := \{h \in G \mid hg = gh, \forall g \in G\}$

$Z(O(n)) = \{\pm \mathbb{1}_n\} \cong \mathbb{Z}_2$

$Z(D_{ever}) = \{R_0, R_{n/2}\} = \mathbb{Z}_3$

<u>Def</u>: $H \subset G$: left-coset $g \cdot H = \{g \cdot h \mid h \in H\} \subset G$

<u>Rem</u>: (i) gH is the rest class of equiv. relation.

$a \sim b :\Leftrightarrow a^{-1}b \in H$.

[reflective, symmetric, transitive]

(ii) g_1H and g_2H are either disjoint or identical.

$g \in g_1H \cap g_2H \Rightarrow g_1h_1 = g = g_2h_2$

$g_1 = g_2h_2h_1^{-1} \in g_2H$

(iii) $g \cdot H$ is a ~~subset~~ subgroup $\Leftrightarrow g \in H$

(iv) $(gH) \longmapsto (gH)^{-1} = Hg^{-1}$ left-coset $\longleftrightarrow$ right coset

Examples:

(i) $G = (\mathbb{Z}, +)$

$H = m\mathbb{Z}$

cosets $m\mathbb{Z}, m\mathbb{Z}+1, m\mathbb{Z}+2, \dots, m\mathbb{Z}+(m+1)$

(ii) $G = D_3$, $H = \mathbb{Z}_3$

$\underset{\overset{\shortparallel}{\{R_0, R_1, R_2\}}}{R_0 \mathbb{Z}_3}$, $S_0\mathbb{Z}_3 = S_1\mathbb{Z}_3 = S_2\mathbb{Z}_3$

Def: $g_1, g_2 \in G$ are conjugate $g_1 \sim g_2$:

$\exists g \in G: g_1 = g g_2 g^{-1}$

This is an equiv. relation:

rest classes $C_g = [g] = \{\tilde{g} g \tilde{g}^{-1} \mid \tilde{g} \in G\}$ Conjugacy classes

Example: D_n: $C_{R_i} = \{R_{i+2j} \mid j \in \mathbb{Z}\}$

$C_{S_i} = \{S_{i+2j} \mid j \in \mathbb{Z}\}$

Def: A subgroup H G is normal $H \triangleleft G$, if it is self conjugate, i.e.,

$gHg^{-1} = H \quad \forall g \in G$

Remark: (i) $H \triangleleft G$, $gH = Hg$ (left = right cosets)

(ii) H is union of conjugacy classes

(iii) $N \subset H \subset G$, $N \triangleleft G \Rightarrow N \triangleleft H$

but in general $N \triangleleft H \triangleleft G \not\Rightarrow N \triangleleft G$ WARNING!

Examples:

(i) center $Z(G)$

(ii) $\varphi: G \to G'$ hom.

$H := \ker(\varphi) = \{g \in G \mid \varphi(g) = e_{G'}\} \triangleleft G$

⌈subgroup: $\varphi(g) = e = \varphi(h)$

$\forall g, h \in H \Rightarrow \varphi(g) \underset{\overset{\shortparallel}{e}}{\cdot} \varphi(h^{-1}) = \varphi(gh^{-1}) \Rightarrow gh^{-1} = H$

normal: $\varphi(ghg^{-1}) = \varphi(g)\underbrace{\varphi(n)}_{e}\varphi(g^{-1}) = \varphi(g)\varphi(g^{-1}) = \varphi(gg^{-1}) = e$

$\Rightarrow ghg^{-1} \in H \ \forall g \in G$ ⌋

e.g. det : $U(n) \to U(1)$ hom.

$\mathrm{Ker}(\det) = \{ A \in U(n) \mid \det(A) = 1 \} = SU(n) \Rightarrow SU(n) \triangleleft U(n)$

(iii) image of normal subgroups are normal in the image of the group.

$H \triangleleft G \Rightarrow \varphi(H) \triangleleft \varphi(G) \subset G'$

$\varphi : G \to G'$

Quotient group:

If $H \triangleleft G \to G/H = \{ gH \mid g \in G \}$ forms a group

$$gH g'H = gg' H \underbrace{(g')^{-1} g'}_{e} H = gg' H \cdot H = gg'H$$

well defined multip. map:

(i) associative

(ii) identity : H

(iii) inverse $(gH)^{-1} = g^{-1}H$

Def: simple groups G don't have normal subgroups (building blocks)

Fact: $\varphi : G \longrightarrow G'$

$H : \ker(\varphi) \triangleleft G$

$G/H = G/\mathrm{kernel}(\varphi) \longrightarrow G'$

$g\ker(\varphi) \longmapsto \varphi(g)$

$f : G/H \xrightarrow{\cong} \varphi(G)$ isomorphism

e.g. $\varphi = \det : U(n) \to U(1)$. $\ker(\varphi) = SU(n)$

$U(n)/SU(n) \cong U(1)$

Product of groups:

Direct product : $G_1 \times G_2$

$(g_1, g_2), (g_1', g_2') \in G_1 \times G_2$

$(g_1, g_2) \cdot (g_1', g_2') = (g_1 g_1', g_2 g_2')$ → satisfy group axioms.

$\pi_2 : G_1 \times G_2 \longrightarrow G_2$ (this is a group homomorfism)

$(g_1, g_2) \longmapsto g_2$

Also true for G_2:

$\mathrm{Ker}(\pi_2) = G_1 \triangleleft G_1 \times G_2$

$G_2 \triangleright$

Fact : $G = N_1 \times N_2 \iff$ (i) $N_1 \triangleleft G$

(ii) $N_1 \cap N_2 = \{e\}$

(iii) $G = N_1 N_2$

Def. : semi-direct product :

N, H groups

$\vartheta : H \longrightarrow \mathrm{Ant}(N) = \mathrm{Iso}(N)$

$N \rtimes_\theta H \underset{\text{as set}}{=} N \times H$

(n,h), $n \in N$, $h \in N$

$$(n,h)\cdot(n',h') = (n\cdot\theta(h)(n'),\ hh')$$

if $\theta =$ identity, then this is the direct product.

$e = (e_N, e_H)$

$(n,h)^{-1} = (\theta(h)^{-1}(n^{-1}),\ h^{-1})$

Check associativity:

$$((n,h)(n',h'))(n'',h'') = (n\,\theta(h)(n'),\ hh')(n'',h'') =$$

$$= (n\,\theta(h)(n')\,\underbrace{\theta(hh')}(n''),\ hh'h'')$$

$$= \theta(h)(\theta(h')(n''))$$

θ is homom.

$$= (n\,\theta(h)(n'\,\theta(h')(n'')),\ hh'h'') =$$

$$= (n,h)((n',h')(n'',h''))$$

Examples: (i) $E(n) = \mathbb{R}^n \rtimes_\theta O(n)$

$$\theta : O(n) \longrightarrow \mathrm{Iso}(\mathbb{R}^n)$$
$$A \longmapsto (a \in \mathbb{R}^n \longmapsto A\cdot a)$$

(ii) $G = D_n$ $\quad N = \mathbb{Z}_n \triangleleft D_n$

$\{R_0, S_0\} = H = \mathbb{Z}_2$

$$\varphi : \mathbb{Z}_n \rtimes \mathbb{Z}_2 \longrightarrow D_4$$
$$(R_i, R_0 = e) \longmapsto R_i R_0 = R_i$$
$$(R_i, S_0) \longmapsto R_i S_0 = S_i$$

$$\begin{aligned}\varphi(R_i, a)\,\varphi(R_j, b) &= R_i\, a\, R_j\, b \\ &= R_i \underbrace{a\, R_j\, a^{-1}}\, a b = \varphi(R_i\, \theta(a)(R_j), ab) \\ R_j' &= a R_j a^{-1} \\ &= \theta(a)(R_j)\end{aligned}$$

$$\theta : \mathbb{Z}_2 \to \mathrm{Iso}(\mathbb{Z}_n)$$
$$a \mapsto (R_i \mapsto a R_i a^{-1})$$

22/04/15

Next lecture: 13.05.15

Problem Set!!!

Sumary of last lecture:

Define basic conceps of groups

- groups
- subgroups
- homomorphisms
- conjugation
- normal groups, quotiont groups
- cosets
- (semi-direct) products