GENERAL RELATIVITY - EXAM SHEET

Newton's axioms: 1. free particles move along straight lines at constant velocity 2. $\vec{F} = m\ddot{\vec{x}}$ 3. actio = −reactio

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Galilei-transformations: universal time $t = t'$ $\forall$ observers; 2 frames S, S' in relative motion $\parallel$ x-axis: $x_1' = x_1 - vt$, $x_2' = x_2$, $x_3' = x_3$
$u_1' = \frac{dx_1'}{dt} = u_1 - v$; $a_1' = \frac{du_1'}{dt} = a_1 \Rightarrow$ accelerations identical in all frames, particles free in S are free in S'

Lorentz transformations: $t \neq t'$, based on two assumptions: 1. spacetime homogeneous 2. $c = c'$ $\forall$ observers
classification of events valid in all frames $ds^2 > 0$ timelike, $ds^2 = 0$ lightlike, $ds^2 < 0$ spacelike

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proper time τ displayed by comoving clock; $ds^2 = c^2 d\tau^2$, $d\tau = \frac{1}{\gamma} dt$; L

(weak) equivalence principle: $m_i = m_g$, i.e. inertial and gravitational mass of any object are equal

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length of a curve C parametrized by λ: $I = \int_C ds = \int_C \sqrt{|g_{\mu\nu} dx^\mu dx^\nu|} = \int_a^b d\lambda \sqrt{\left|g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda}\right|}$, area: $dA = \sqrt{|g_{\mu\mu}|} dx^\mu \overset{2d}{=} \sqrt{|g_{11} g_{22}|} dx^1 dx^2$

Riemannian manifolds are locally Cartesian, i.e. can locally be matched to Minkowski space

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Christoffel symbols (affine connection): $\Gamma^\rho_{\mu\nu} = \frac{1}{2} g^{\rho\sigma} (\partial_\mu g_{\nu\sigma} + \partial_\nu g_{\sigma\mu} - \partial_\sigma g_{\mu\nu})$ can be obtained by diff. $g_{\mu\nu} = \vec{e}_\mu \cdot \vec{e}_\nu$ w.r.t. x^ρ
and cyclically perm. indices; $\Gamma^\rho_{\mu\nu} = \vec{e}^\rho \frac{\partial}{\partial x^\nu} \vec{e}_\mu$; equivalence: locally Cartesian $\longleftrightarrow$ vanishing connection; torsion $T^\alpha_{\mu\nu} := \Gamma^\alpha_{\mu\nu} - \Gamma^\alpha_{\nu\mu}$

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cov. derivative of contravector $\nabla_\mu V^\nu = \partial_\mu V^\nu + \Gamma^\nu_{\mu\rho} V^\rho$, of covector $\nabla_\mu V_\nu = \partial_\mu V_\nu - \Gamma^\rho_{\mu\nu} V_\rho$, $\nabla_\mu V^\nu = \partial_\mu V^\nu$ if locally Cartesian

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geodesic (on torsion-free manifold): shortest connection of two points, more generally curve with constant tangent
geodesic equation (eq. of parallel transport): $\frac{d^2 x^\mu}{d^2\lambda} + \Gamma^\mu_{\rho\sigma} \frac{dx^\rho}{d\lambda} \frac{dx^\sigma}{d\lambda} = 0 = \frac{du^\mu}{d\lambda} + \Gamma^\mu_{\rho\sigma} u^\rho u^\sigma$, path-dependent due to varying $\Gamma^\mu_{\rho\sigma}$

Riemann-tensor $R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda} \Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda} \Gamma^\lambda_{\mu\sigma}$, $R_{\rho\sigma\mu\nu} = g_{\rho\lambda} R^\lambda{}_{\sigma\mu\nu}$; symmetries: $R_{\rho\sigma\mu\nu} = -R_{\sigma\rho\mu\nu} = R_{\sigma\rho\nu\mu}$
$R_{\rho\sigma\mu\nu} = R_{\mu\nu\rho\sigma}$, $R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0$, $d = \frac{n^2}{12}(n^2 - 1)$ indep. comp.; metric's comp. constant $\Longleftrightarrow R^\rho{}_{\sigma\mu\nu} = 0 \Longleftrightarrow$ space flat

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Ricci-tensor $R_{\mu\nu} = R^\lambda{}_{\mu\lambda\nu} = R_{\nu\mu}$, Ricci-scalar $R = R^\mu{}_\mu = g^{\mu\nu} R_{\mu\nu}$

Bianchi-identity $\nabla_\lambda R_{\rho\sigma\mu\nu} + \nabla_\rho R_{\sigma\lambda\nu} + \nabla_\sigma R_{\lambda\rho\mu\nu} = 0$; Riemann-tensor on 2-sphere $R^i{}_{jkm} = \frac{1}{r^2}(\delta^i_k g_{jm} - \delta^i_m g_{jk})$

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energy-mom. tensor of perfect fluid (:= isotropic in its rest frame) $T^{\mu\nu} = (\rho + \frac{p}{c^2}) u^\mu u^\nu - p\eta^{\mu\nu} = \mathrm{diag}(\rho/c^2, p, p, p)$

Einstein field eqs. $G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$

FLRW: $ds^2 = c^2 dt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\varphi^2\right]$, inv. under $k \to k/|k|$, $r \to \sqrt{|k|} r$, $a \to a/\sqrt{k}$; metric singular at $r = \pm 1$

$k = -1 \longleftrightarrow$ negative curvature $\longleftrightarrow$ open universe $\longleftrightarrow \rho < \rho_{crit} \longleftrightarrow \Omega < 1$ — for $k = \pm 1 \to$ new radial coordinate $r = \sin(h)\chi$, $\frac{dr}{d\chi} = \sqrt{1 \mp r^2}$
$k = 0 \longleftrightarrow$ no curvature $\longleftrightarrow$ flat universe $\longleftrightarrow \rho = \rho_{crit} \longleftrightarrow \Omega = 1$ — $\rho_{crit} = \frac{3H^2}{8\pi G}$
$k = 1 \longleftrightarrow$ positive curvature $\longleftrightarrow$ closed universe $\longleftrightarrow \rho > \rho_{crit} \longleftrightarrow \Omega > 1$ — $\Gamma^1_{11} = \frac{kr}{1-kr^2}$

$\Gamma^0_{11} = \frac{a\dot{a}}{c(1-kr^2)}$, $\Gamma^0_{22} = \frac{a\dot{a}r^2}{c}$, $\Gamma^0_{33} = \frac{a\dot{a}r^2}{c}\sin^2\theta$, $\Gamma^1_{01} = \Gamma^1_{10} = \Gamma^2_{02} = \Gamma^2_{20} = \Gamma^3_{30} = \Gamma^3_{03} = \frac{\dot{a}}{ca}$, $\Gamma^1_{22} = -r(1-kr^2)$, $\Gamma^1_{33} = \Gamma^1_{22}\sin^2\theta$, $\Gamma^2_{33} = -\sin\theta\cos\theta$
$\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}$, $\Gamma^3_{32} = \cot\theta$; $R_{00} = 3\frac{\ddot{a}}{c^2 a}$, $R_{11} = -\frac{1}{c^2}\frac{a\ddot{a} + 2\dot{a}^2 + 2c^2k}{1-kr^2}$, $R_{22} = -\frac{r^2}{c^2}\frac{a\ddot{a} + 2\dot{a}^2 + 2c^2k}{1}$, $R_{33} = R_{22}\sin^2\theta$, $R = \frac{6}{c^2a^2}(a\ddot{a} + \dot{a}^2 + c^2k)$

Friedmann eqs.: $H^2 = \frac{\dot{a}^2}{a^2} = \frac{8}{3}\pi G\rho + \frac{c^2}{3}\Lambda - \frac{c^2}{a^2}k$, $\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + \frac{3}{c^2}p) + \frac{c^2}{3}\Lambda$; eq. of state $\omega = p/\rho c^2$

Schwarzschild: $ds^2 = -(1 - \frac{2GM}{r})dt^2 + \frac{dr^2}{1 - \frac{2GM}{r}} + r^2 d\Omega^2$, contains two symmetry assumptions: sphericity & static
asymptotically flat for $r \to \infty$ or $M \to 0$; g_{tt} and g_{rr} switch sign at $r = r_s = \frac{2GM}{c^2} \Rightarrow$ light cones flip and are undefined at r_s
$\Gamma^1_{00} = \frac{GM}{r^3}(r - 2GM)$, $\Gamma^1_{11} = \frac{-GM}{r(r-2GM)} = -\Gamma^0_{01}$, $\Gamma^2_{12} = \Gamma^2_{21} = \Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r}$, $\Gamma^1_{22} = -r(1 - \frac{2GM}{r})$, $\Gamma^1_{33} = \Gamma^1_{22}\sin^2\theta$, $\Gamma^2_{33} = -\sin\theta\cos\theta$, $\Gamma^3_{23} = \cot\theta$
$R^{\mu\nu\rho\sigma} R_{\mu\nu\rho\sigma} = \frac{3r_s^2}{r^6} = R \to$ curvature finite except at $r = 0$; in Kruskal coord. (u, v, θ, ϕ): $ds^2 = \frac{4r_s^3}{r} e^{-\frac{r}{2GM}}(dv^2 - du^2) + r^2 d\Omega^2$

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linearized gravity: $\Box \bar{h}^{\mu\nu} = -\frac{16\pi G}{c^4} T^{\mu\nu}$, gauge: $\partial_\nu \bar{h}^{\mu\nu} = 0$, $\bar{h}^{\mu\nu} = h^{\mu\nu} - \frac{h}{2}\eta^{\mu\nu}$; in vacuum: $\Box \bar{h}^{\mu\nu} = 0$ solved by plane wave
$\bar{h}^{\mu\nu} = A^{\mu\nu} e^{\pm i k_\sigma x^\sigma}$ with $k^2 = 0$; traceless transverse gauge fixes all gauge freedom, leaves 2 physical d.o.f., $\bar{h}^{TT}_{\mu\nu} = h^{TT}_{\mu\nu}$, $A^{\mu\nu}_{TT} = \begin{pmatrix} 0 & & & 0 \\ & a & b & \\ & b & -a & \\ 0 & & & 0 \end{pmatrix}$
harmonic gauge condition $\partial_\mu \bar{h}^\mu{}_\nu = 0$

Lie derivative of V w.r.t. vector field a: $\mathcal{L}_a V^\mu = -a^\nu \partial^\mu V_\nu + V^\nu \partial_\nu a^\mu$, $\mathcal{L}_a V_\mu = a^\nu \partial_\nu V_\mu + V_\nu \partial_\mu a^\nu$
$\mathcal{L}_a g_{\mu\nu} = \nabla_\nu a_\mu + \nabla_\mu a_\nu$ for metric compatible connection, if $\mathcal{L}_\xi g_{\mu\nu} = 0$, then ξ is Killing vector field

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Killing vector fields are vector fields on a Riemannian manifold that preserve the metric. They are the infinitesimal generators of isometries which in turn generate symmetries.

Useful formulae from exersises

- harmonic oscillator $m\ddot{x} = -kx$ with angular frequency $\omega = \sqrt{\frac{k}{m}} = \frac{2\pi}{T}$ and period T, $x(t) = A\cos(\omega t + \phi)$, potential energy $U = \frac{1}{2}kx^2(t)$
- gravit. force $F_g = -\frac{Gm_1m_2}{r^2}$, centrifugal force $F_c = \frac{m_2v^2}{r}$; in stable orbit $-F_g = F_c \Longrightarrow$ orbital velocity $v = \sqrt{\frac{Gm_1}{r}}$
- ~~Lagrangian~~ e.o.m. invariant under $L(q,\dot{q},t) \longrightarrow \alpha L(q,\dot{q},t) + \frac{d}{dt}f(q,t)$, this yields symmetries and conservation laws, $\frac{\partial L}{\partial q_i} = 0 \Rightarrow \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{d}{dt}p_i = 0$
- escape velocity follows from $T + V = \frac{m}{2}v_{esc}^2 - \frac{GmM}{r^2} = 0 \Longrightarrow v_{esc} = \sqrt{\frac{2GM}{r}}$, for $v_{esc} = c$, we get $R_s = \frac{2GM}{c^2}$
- $\eta_{\mu\nu} = \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \eta_{\rho\sigma}$ follows from requiring $|x'|^2 = |x|^2$
- relativistic velocity addition: $w = \frac{u+v}{1+\frac{uv}{c^2}}$; proper velocity $v = \frac{\delta}{\tau} = \frac{\frac{1}{\gamma}d}{\tau} = \sqrt{1-\frac{v^2}{c^2}}\,\frac{d}{\tau}$, where e.g. $\frac{d}{\tau} = c$ for $d = x$ ly and $\tau = x$ years

  relativistic Doppler effet: observer moving away with v measures frequency $f' = \sqrt{\frac{1-\beta}{1+\beta}}\,f_0$

  two events separated by timelike (spacelike) intervals, i.e. $\Delta s^2 \underset{(<)}{>} 0$, may have (may not have) cause-effect relationship, $\exists$ frame s.t. events at same location (time)
- weak field limit: hdds for slowly moving particles ($v \ll c$) and weak gravitational fields $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$ with $|h_{\mu\nu}| \ll 1$ (that are static, i.e. $\partial_t g_{\mu\nu} = 0$)
- Christoffel symbds in spherical coord.: $\Gamma^1{}_{22} = -r$, $\Gamma^1{}_{33} = -r\sin^2\theta$, $\Gamma^2{}_{12} = \Gamma^2{}_{21} = \frac{1}{r}$, $\Gamma^2{}_{33} = -\sin\theta\cos\theta$, $\Gamma^3{}_{13} = \Gamma^3{}_{31} = \frac{1}{r}$, $\Gamma^3{}_{23} = \Gamma^3{}_{32} = \cot\theta$, all others zero
- effective potential for orbiting objects $V_{eff}(r) = -\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{c^2r^3}$, where last term is a pure GR contribution
- particles experiencing grav. waves don't change their position because of them
- Pauli matrices $\sigma_1 = \begin{pmatrix}0&1\\1&0\end{pmatrix}$, $\sigma_2 = \begin{pmatrix}0&-i\\i&0\end{pmatrix}$, $\sigma_3 = \begin{pmatrix}1&0\\0&-1\end{pmatrix}$
- coordinate transformations: Cartesian $\longleftrightarrow$ spherical $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}r\cos\phi\sin\theta\\r\sin\phi\sin\theta\\r\cos\theta\end{pmatrix}$, $\begin{pmatrix}r\\\theta\\\phi\end{pmatrix} = \begin{pmatrix}\sqrt{x^2+y^2+z^2}\\\arccos(z/\sqrt{x^2+y^2+z^2})\\\arctan(y/x)\end{pmatrix}$; Cartesian $\longleftrightarrow$ cylindrical $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}\rho\cos\phi\\\rho\sin\phi\\z\end{pmatrix}$, $\begin{pmatrix}\rho\\\phi\\z\end{pmatrix} = \begin{pmatrix}\sqrt{x^2+y^2}\\\arcsin(y/\rho)\\z\end{pmatrix}$

  Cartesian $\longleftrightarrow$ ellipsoidal $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}(r^2+a^2)^{\frac{1}{2}}\sin\theta\cos\phi\\(r^2+a^2)^{\frac{1}{2}}\sin\theta\sin\phi\\r\cos\theta\end{pmatrix}$, Cartesian $\longleftrightarrow$ ellipsoid $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}a\cosh\mu\cos\nu\\a\sinh\mu\sin\nu\end{pmatrix}$; Cartesian $\longleftrightarrow$ hyperbolic $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}ve^u\\ve^{-u}\end{pmatrix}$, $\begin{pmatrix}u\\v\end{pmatrix} = \begin{pmatrix}\ln(x/y)\\\sqrt{xy}\end{pmatrix}$
- Tensor $\underline{T}$ of rank (k,l) is a multilinear map that takes k dual vectors and l vectors and projects them onto a number in $\mathbb{R}$.

  Expansion: $\underline{T} = T^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_l}\,\partial_{\mu_1}\otimes\cdots\otimes\partial_{\mu_k}\otimes dx^{\nu_1}\otimes\cdots\otimes dx^{\nu_l}$. Transformation: $T^{\mu_1'\ldots\mu_k'}{}_{\nu_1'\ldots\nu_l'} = \frac{\partial x^{\mu_1'}}{\partial x^{\mu_1}}\cdots\frac{\partial x^{\mu_k'}}{\partial x^{\mu_k}}\frac{\partial x^{\nu_1}}{\partial x^{\nu_1'}}\cdots\frac{\partial x^{\nu_l}}{\partial x^{\nu_l'}}T^{\mu_1\ldots\mu_k}{}_{\nu_1\ldots\nu_l}$

  Christoffel symbol transformation $\Gamma^{\alpha'}{}_{\mu'\nu'} = \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\frac{\partial x^{\alpha'}}{\partial x^\alpha}\Gamma^\alpha{}_{\mu\nu} - \frac{\partial x^\mu}{\partial x^{\mu'}}\frac{\partial x^\nu}{\partial x^{\nu'}}\frac{\partial^2 x^{\alpha'}}{\partial x^\mu\partial x^\nu}$