Quantum Field Theory II - Exam Sheet

Path Integral

- QM transition amplitude: $\langle q_F, t_F | q_I, t_I\rangle = \langle q_F | e^{-i\hat{H}\overbrace{(t_F - t_I)}^{\Delta t}} | q_I\rangle$, $\delta t = \frac{\Delta t}{N+1}$, $N \to \infty$, $e^{-i\hat{H}\Delta t} = e^{-i\hat{H}\delta t} \dots e^{-i\hat{H}\delta t}$
  insert $1 = \int dq_k |q_k\rangle\langle q_k|$ to get $\langle q_F, t_F | q_I, t_I\rangle = \lim_{N\to\infty} \int \prod_{k=1}^{N} dq_k \langle q_F | e^{-i\hat{H}\delta t} | q_N\rangle\langle q_N | e^{-i\hat{H}\delta t} | q_{N-1}\rangle \dots \langle q_1 | e^{-i\hat{H}\delta t} | q_I\rangle$, now assume $\hat{H}(\hat{p},\hat{q}) = f(\hat{p}) + V(\hat{q})$
  and use BCH-formula $e^{-i\hat{H}\delta t} = e^{-iV(\hat{q})\delta t} e^{-if(\hat{p})\delta t} + \mathcal{O}(\delta t^2)$, thus $\langle q_{k+1} | e^{-i\hat{H}\delta t} | q_k\rangle = \int p_k \langle q_{k+1} | e^{-if(\hat{p})\delta t} | p_k\rangle\langle p_k | e^{-iV(\hat{q})\delta t} | q_k\rangle$, where
  $\langle p_k | q_k\rangle = \frac{e^{-ip_k q_k}}{\sqrt{2\pi}}$ so that $\int p_k e^{-i(f(\hat{p}) + V(\hat{q}))\delta t} e^{ip_k(q_{k+1} - q_k)}$. Reinsertion gives $\langle q_F, t_F | q_I, t_I\rangle = \lim_{N\to\infty} \int \frac{dp_0}{2\pi} \prod_{k=1}^{N} \frac{dp_k dq_k}{2\pi} e^{i\sum_{k=0}^{N}\left[p_k \frac{q_{k+1} - q_k}{\delta t} - H(p_k, q_k)\right]\delta t}$
  $\xrightarrow{\delta t \to 0} \int_{q_I}^{q_F} \mathcal{D}q(t)\mathcal{D}p(t) e^{i\int_{t_I}^{t_F}(p\dot{q} - H(\hat{p},\hat{q}))dt} = \int_{q_I}^{q_F} \mathcal{D}q(t)\mathcal{D}p(t) \underbrace{e^{i\int_{t_I}^{t_F} dt\, L(p,q)}}_{e^{iS[p,q]}}$
- Scalar fields: path integral master formula for quantum correlation fcts. $G(x_1, \dots, x_n) = \langle\Omega| T \prod_{i=1}^{n} \hat{\phi}_H(x_i) |\Omega\rangle = \lim_{T\to\infty(1-i\epsilon)} \frac{\int \mathcal{D}\phi(x) \prod_{i=1}^{n} \phi(x_n) e^{i\int_{-T}^{T} d^4x\, \mathcal{L}(\phi(x))}}{\int \mathcal{D}\phi(x)\, e^{i\int_{-T}^{T} d^4x\, \mathcal{L}(\phi(x))}}$
- $Z[J]$ generating functional of Green's fcts.: $G(x_1, \dots, x_n) = \frac{1}{Z[0]} \prod_{i=1}^{n} \frac{\delta}{i\delta J(x_i)} Z[J]\Big|_{J=0}$; $Z_0[J] = \int \mathcal{D}\phi\, e^{-\frac{1}{2}\phi \cdot D_F^{-1} \phi + i\phi \cdot J}$
- Schwinger-Dyson eq. $\int \mathcal{D}\phi \left(\frac{\delta S}{\delta\phi(x)} + J(x)\right) e^{i(S[\phi] + J\cdot\phi)} = 0$, states that cl. e.o.m. holds as operator eq. in quantum theory; $\left(\frac{\delta S}{\delta\phi}\Big|_{\phi = \frac{\delta}{i\delta J}} + J\right) Z[J] = 0$
- effective action $iW[J]$ generating fctnal of fully connected Green's fcts.; $iW[J] = \ln\frac{Z[J]}{Z[0]}$
- 1PI effective action $\Gamma[\varphi]$ Legendre transform of $W[J]$, i.e. $\Gamma[\varphi] := W[J_\varphi] - \varphi \cdot J_\varphi$, $\frac{\delta\Gamma}{\delta\varphi(x)} = -J_\varphi(x)$, $G_2^{(c)}(x_1, x_2) = -\Gamma_2^{-1}(x_1, x_2)$
- fermionic path integral: $\langle\Psi_F(\vec{x}_F), t_F | \Psi_I(\vec{x}_I), t_I\rangle = \int \mathcal{D}\bar{\Psi}(x)\mathcal{D}\Psi(x)\, e^{i\int_{t_I}^{t_F} d^4x\, \mathcal{L}(\Psi, \bar{\Psi})}$
- if $H(p,q) = \frac{p^2}{2m} + V(q)$, integrate $\int \mathcal{D}p$ explicitly by analytic continuation $\delta t \to \delta t(1 - i\epsilon)$ (so that $\mathrm{Re}(a) > 0$) to get $\int \frac{dp_k}{2\pi} e^{i(p_k(q_{k+1} - q_k) - \delta t \frac{p_k^2}{2m})} = e^{i\frac{m}{2\delta t}(q_{j+1} - q_j)^2} \sqrt{\frac{-im}{2\pi\delta t}}$
- ex. of compl. the square in fermionic path integral: $S_0[\bar{\Psi}, \Psi] + \bar{\eta}\cdot\Psi + \bar{\Psi}\cdot\eta = i\bar{\Psi}\cdot S_F^{-1}\cdot\Psi + \bar{\eta}\cdot\Psi + \bar{\Psi}\cdot\eta = (\bar{\Psi} - iS_F\bar{\eta}) \cdot iS_F^{-1} \cdot (\Psi - iS_F\eta) + i\bar{\eta}\cdot S_F\cdot\eta$

Renormalization

- Superficial degree of divergence D: ϕ^n in d dim $D_{\phi^n} = d - \overbrace{\left(d - n\frac{d-2}{2}\right)}^{[\lambda^{(n)}]} V - \frac{d-2}{2} E$ with $nV = 2I + E$ and Euler's formula $L = I - V + 1$ or $L = \sum_i I_i - \sum_j V_j + 1$
- $D \geq 0$ may still be finite due to symmetries, $D < 0$ may still be divergent due to divergent subdiagram, tree-level diagrams have $D = 0$ but are finite
- a QFT is 1. renormalizable if number of superf. diverg. ampl. is finite, but superf. divergs. appear at every order; 2. super-renormalizable if number of superf. divergent ampl. is finite 3. non-renormalizable if infinite; 1. $\hat{=} [\lambda] = 0$, 2. $\hat{=} [\lambda] > 0$, 3. $\hat{=} [\lambda] < 0$
- BPHZ theorem: if a QFT is renormalizable, i.e. has finite number of divergent diagr., one can absorb divergencies order by order in counterterms
- in a QFT with dimensionless coupling, $[\lambda] = 0$, the dimension of an amplitude is equal to D
- QED: $D_{QED} = 4 - E_\gamma - \frac{3}{2}E_f$, QED is renormalizable since $[e] = 0$, symmetric under charge conj.!, respects Ward identity!
- Callan-Symanzik eq.: $\mu\frac{d}{d\mu} G_n^{(0)}(x_i; \lambda_0, m_0)\Big|_{\lambda_0, m_0 \text{ fixed}} = 0 = \mu\frac{d}{d\mu}\left(Z^{n/2} G_n(x_i; \lambda, m; \mu)\right)\Big|_{\lambda_0, m_0 \text{ fixed}}$, where $G_n(x_1, \dots, x_n) = \langle\Omega| T \prod_{i=1}^{n} \phi(x_i) |\Omega\rangle\big|_{\text{connected}}$ and $\phi(x_i) = Z^{-\frac{1}{2}} \phi_0(x_i)$; chain rule yields $\left(\mu\frac{\partial}{\partial\mu} + \beta_\lambda\frac{\partial}{\partial\lambda} + \beta_{m^2}\frac{\partial}{\partial m^2} + n\cdot\gamma_\phi\right) G_n(x_i; \lambda, m; \mu) = 0$ with $\beta_\lambda := \mu\frac{d\lambda}{d\mu}\Big|_{\lambda_0, m_0}$, $\beta_{m^2} = \mu\frac{dm^2}{d\mu}\Big|_{\lambda_0, m_0}$, $\gamma_\phi = \frac{1}{2}\frac{\mu}{Z}\frac{dZ}{d\mu}\Big|_{\lambda_0, m_0}$
- Renormalization group (RG) flow (eq.) given by the beta-function $\beta(\lambda) = \frac{d\lambda(\mu)}{d\ln(\mu)}$ describes change of $\lambda(\mu)$ with μ: $\beta(\lambda) \underset{(<)}{>} 0 \Leftrightarrow \lambda(\mu)$ increases (decreases) as μ increases
  Landau pole: $\lambda \to \infty$ as $\mu \to \mu^*$ with μ^* finite; Gaussian IR (UV) fixed point: $\lambda \to 0$ as $\mu \to 0$ (∞); $\beta = 0\ \forall\mu \Rightarrow \lambda(\mu) \Leftrightarrow$ theory is scale-ind. or conformal
- RG flow of dimensionful operators: $\int d^dx\, C_i O_{d_i}$ with $[O_{d_i}] = d_i$, $[C_i] = d - d_i$, $C_i = g_i\mu^{d-d_i}$, then CS eq. reads $0 = \left[\mu\frac{\partial}{\partial\mu} + \beta_\lambda\frac{\partial}{\partial\lambda} + n\gamma_\phi + \underbrace{(\gamma_i + d_i - d)g_i}_{\beta_i = \frac{dg_i}{d\ln\mu}}\frac{\partial}{\partial g_i}\right] G_n(x_i; \lambda, g_i)$
  if $[C_i] = d - d_i \underset{(>)}{<} 0 \Leftrightarrow C_i$ is non-renormalizable (super-renormalizable) coupling which becomes irrelevant (relevant) in the IR
- Wilsonian approach: QFT just an effective description of physics accurate only at energies below an intrinsic cutoff Λ_0 (or at large distances)
- Dim. reg. integrals: $\int\frac{d^dL}{(2\pi)^d}\frac{1}{(L^2 - \Delta)^n} = \frac{(-1)^n i}{(4\pi)^{d/2}}\frac{\Gamma(n - \frac{d}{2})}{\Gamma(n)}\Delta^{\frac{d}{2} - n}$, $\int\frac{d^dL}{(2\pi)^d}\frac{L^2}{(L^2 - \Delta)^n} = \frac{(-1)^{n-1} i}{(4\pi)^{d/2}}\frac{d}{2}\frac{\Gamma(n - \frac{d}{2} - 1)}{\Gamma(n)}\Delta^{1 + \frac{d}{2} - n}$, $\int\frac{d^dL}{(2\pi)^d}\frac{L^\mu L^\nu}{(L^2 - \Delta)^n} = \frac{(-1)^{n-1} i}{(4\pi)^{d/2}}\frac{\eta^{\mu\nu}}{2}\frac{\Gamma(n - \frac{d}{2} - 1)}{\Gamma(n)}\Delta^{1 + \frac{d}{2} - n}$
  frequent identity: $\frac{\Gamma(2 - \frac{d}{2})}{(4\pi)^{d/2}}\Delta^{\frac{d}{2} - 2} = \frac{1}{16\pi^2}\left(\frac{2}{\epsilon} - \ln\Delta - \gamma + \ln 4\pi + \mathcal{O}(\epsilon)\right)$, $\frac{1}{A_1^{m_1} A_2^{m_2} \dots A_n^{m_n}} = \int_0^1 dx_1 \dots dx_n\, \delta\left(\sum_i x_i - 1\right)\frac{\prod x_i^{m_i - 1}}{\left[\sum_i x_i A_i\right]^{\sum m_i}}\frac{\Gamma(m_1 + \dots + m_n)}{\Gamma(m_1)\dots\Gamma(m_n)}$, $\frac{1}{AB^n} = \int_0^1 dx\, \frac{n(1-x)^{n-1}}{[xA + (1-x)B]^{n+1}}$
- Trace identities: $\mathrm{tr}(\gamma^\mu) = 0$, $\mathrm{tr}(\gamma^{\mu_1}\dots\gamma^{\mu_n}) = 0$ for n odd, $\mathrm{tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}$, $\mathrm{tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(\eta^{\mu\nu}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho})$, $\gamma^\mu\gamma^\nu\gamma_\mu = -2\gamma^\nu$, $\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\mu = 4\eta^{\nu\rho}$

Yang-Mills theory

- starting point: Lie group H with Lie algebra $Lie(H)$ s.t. every $h \in H$ can be expressed as $h = e^{-ig\alpha}$ with $g \in \mathbb{R}$, $\alpha = \alpha_a T^a \in Lie(H)$, where T^a are a basis of $Lie(H)$, i.e. the generators of H, satisfying defining relation $[T^a, T^b] = i f^{abc} T^c$ and Jacobi-identity $[[T^a, T^b], T^c] + c.p. =$
- (adjoint) covariant derivative: $D_\mu \alpha(x) := \partial_\mu \alpha(x) + ig[A_\mu(x), \alpha(x)]$ used to define $F_{\mu\nu}(x) = \frac{1}{ig}[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu]$, field strength satisfies Bianchi identity $D_\mu F_{\alpha\beta} + D_\alpha F_{\beta\mu} + D_\beta F_{\mu\alpha} = 0$
- pure Yang-Mills Lagrangian: $\mathcal{L}_{YM} = -\frac{1}{2} tr(F_{\mu\nu} F^{\mu\nu}) = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu a}$ is gauge invariant; e.o.m. for A_μ: $D_\mu F^{\mu\nu} = 0 = \partial_\mu F^{\mu\nu} + ig[A_\mu, F^{\mu\nu}]$
- generators T^a normalized to $Tr(T^a T^b) = \frac{1}{2}\delta^{ab}$
- for $\mathcal{L}_{YM} + \mathcal{L}_{matter}$, the e.o.m. become $D_\mu F^{\mu\nu a} = j^{\nu a}$ for some $j^{\nu a}(\phi, \psi, \ldots)$

Possible exam questions

1. Derivation of the fermionic PI: trans. ampl. $\langle \psi_F, t_F | \psi_I, t_I \rangle = \langle \psi_F | e^{-i\hat{H}(t_F - t_I)} | \psi_I \rangle$ with $e^{-i\hat{H}(t_F - t_I)} = \lim_{N\to\infty} \left(e^{-i\hat{H}\delta t}\right)^N$, $\delta_t = \frac{t_F - t_I}{N+1}$. Now insert identities $\langle \psi_F, t_F | \psi_I, t_I \rangle = \lim_{N\to\infty} \int \prod_{j=1}^N d\psi_j^* d\psi_j \langle \psi_F | e^{-i\hat{H}\delta t} | \psi_N \rangle \ldots \langle \psi_1 | e^{-i\hat{H}\delta t} | \psi_I \rangle$ since $\mathbb{1} = \int d\psi_j^* d\psi_j |\psi_j\rangle e^{-\psi_j^* \psi_j} \langle \psi_j |$. We assume $\hat{H} = \hat{\psi}^\dagger M \hat{\psi}$ with Grassmann even M. Then $\langle \psi_{j+1} | e^{-i\hat{H}\delta t} | \psi_j \rangle = \langle \psi_{j+1} | (1 - i\hat{\psi}^\dagger M \hat{\psi}\delta t) | \psi_j \rangle = e^{\psi_{j+1}^* \psi_j} e^{-i\psi_{j+1}^* M \psi_j \delta t} = e^{\psi_{j+1}^* \psi_{j+1}} e^{\psi_{j+1}^*(\psi_j - \psi_{j+1})} e^{-i\psi_{j+1}^* M \psi_j \delta t}$, where the factors $e^{\psi_{j+1}^* \psi_{j+1}}$ cancel with those from identities, except for $e^{\psi_F^* \psi_F}$. Thus $\langle \psi_F, t_F | \psi_I, t_I \rangle = \lim_{N\to\infty} \int \prod_{j=1}^N d\psi_j^* d\psi_j\, e^{\psi_F^* \psi_F} e^{\sum_{j=0}^N \left(-\psi_{j+1}^*(\psi_{j+1} - \psi_j) - i\delta t\, H(\psi_{j+1}^*, \psi_j)\right)}$ $\xrightarrow{N\to\infty} \int D\psi^* D\psi\, e^{i\int_{t_I}^{t_F} dt \left(\psi^* i \frac{\partial\psi}{\partial t} - H\right)}$.

2. Renormalization of QED: $D = 4L - P_e - 2P_\gamma$, $L = P_e + P_\gamma - V + 1$, $V = 2P_\gamma + I_\gamma = \frac{1}{2}(2P_e + I_e) \implies D = 4 - \frac{3}{2} I_e - I_\gamma$

   Divergent amplitudes (superficially): $D=4$ can be absorbed into V_0, $D=3$ zero by $A_\mu \to -A_\mu$-symmetry, $\boxed{D=2}$ logarithmically divergent, $D=1$ 0 by $A_\mu \to -A_\mu$, $D=0$ divergent parts cancel due to Ward identity, 0 by charge conserv., $\boxed{D=1}$ logarithm. divergent, $\boxed{D=0}$ logarithm. divergent, $\mathcal{L}_{QED} = -\frac{1}{4}(F_{\mu\nu})^2 + \bar{\psi}(i\not{\partial} - m)\psi - eA_\mu \bar{\psi}\gamma^\mu\psi - \frac{1}{4}\delta_A (F_{\mu\nu})^2 + \bar{\psi}(i\delta_\psi \not{\partial} - \delta_m)\psi - \delta_e A_\mu \bar{\psi}\gamma^\mu\psi$, where $\psi = Z_\psi^{-\frac{1}{2}} \psi_0$, etc.

   Feynman rules: $\mu \xrightarrow{q} \nu = \frac{-i\eta^{\mu\nu}}{q^2}$, $\xrightarrow{p} = \frac{i(\not{p} + m)}{p^2 - m^2}$, $= -ie\gamma^\mu$, $\mu \xrightarrow{q} \nu = -i(\eta_{\mu\nu} q^2 - q_\mu q_\nu)\delta_A$, $\xrightarrow{p} = i(\not{p}\delta_\psi - \delta_m)$, $\mu = -ie\gamma^\mu \delta_e$

   Renormalization conditions: $\Sigma(\not{p} = m) = 0$, $\frac{d}{d\not{p}} \Sigma(\not{p} = m) = 0$, $\Pi(q^2 = 0) = 0$, $-ie\Gamma^\mu(p' - p = 0) = -ie\gamma^\mu$, where $\mu \xrightarrow{q}$ 1PI $\nu = i\Pi^{\mu\nu}(q) = i(q^2\eta^{\mu\nu} - q^\mu q^\nu)\Pi(q^2)$ and $(\ \ \mu)_{amp} = -ie\Gamma^\mu(p', p)$. We use dim. reg. to control UV and photon mass μ to control IR divergencies.

$$\Big|_{1\text{-loop}} = \xrightarrow{p} + \ (p,\ k,\ p-k) \ + \ \xrightarrow{p} = \frac{i(\not{p} + m)}{p^2 - m^2} - i\Sigma_2(p) + i(\not{p}\delta_\psi - \delta_m)$$

$$-i\Sigma_2(p) = (-ie)^2 \int \frac{d^dk}{(2\pi)^d} \gamma^\mu \frac{i(\not{k} + m)}{k^2 - m^2} \gamma_\mu \frac{-i}{(p-k)^2 - \mu^2} = -e^2 \int_0^1 dx \int \frac{d^dl}{(2\pi)^d} \frac{-2x\not{p} + 4m}{[l^2 - \Delta]^2} = \frac{-ie^2}{(4\pi)^{d/2}} \int_0^1 dx \frac{\Gamma(\frac{\epsilon}{2})\left[(4-\epsilon)m - 2(1-\frac{\epsilon}{2})x\not{p}\right]}{\left[(1-x)m^2 + x\mu^2 - x(1-x)p^2\right]^{\epsilon/2}}$$

$$\Sigma(\not{p} = m)\big|_{1\text{-loop}} = -i\Sigma_2(m) + i(m\delta_\psi - \delta_m) \stackrel{!}{=} 0 \implies m\delta_\psi - \delta_m = \Sigma_2(m) = \frac{e^2 m}{(4\pi)^{d/2}} \int_0^1 dx \frac{\Gamma(\frac{\epsilon}{2})(4 - 2x - \epsilon(1-x))}{\left[(1-x)^2 m^2 + x\mu^2\right]^{\epsilon/2}}$$

$$\frac{d}{d\not{p}} \Sigma(\not{p} = m)\big|_{1\text{-loop}} = -i\frac{d}{d\not{p}}\Sigma_2(m) + i\delta_\psi \stackrel{!}{=} 0 \implies \delta_\psi = -\frac{e^2}{(4\pi)^{d/2}} \int_0^1 dx \frac{\Gamma(\epsilon/2)}{\left[(1-x)^2 m^2 + x\mu^2\right]^{\epsilon/2}} \left[(2-\epsilon)x - \frac{\epsilon}{2} \frac{2x(1-x)m^2}{(1-x)^2 m^2 + x\mu^2}\left(4 - 2x - \epsilon(1-x)\right)\right]$$