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General Relativity (MKTP3) Summer Term 2015

Exercise sheet 11 23 June 2015

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1. (15 points) h_{0i} **again!**
On the last sheet, we got that a constantly rotating ball of radius R and mass M gives us to first order not only a quasi-newtonian term

$$h_{\mu\mu} = -\frac{2\Phi(r)}{c^2},$$

(where $\Phi(r) = GM/r$), but also something related to the angular velocity ω, namely

$$h_{0i} = -\frac{4GMR^2(\epsilon_{ijk}\omega^j x^k)}{5c^3r^3}.$$

Both terms depend on the distance to the ball's centre r.
Let us now see how geodesics are affected by this. For this approximation, we're going to set $\mathrm{d}t = \mathrm{d}\tau$ and $(u^\mu) = (c, v^i)$.

(a) Reason why the Christoffel symbol Γ can be written as

$$2\Gamma^\alpha_{\mu\nu} = \eta^{\alpha\sigma}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}).$$

in the weak-field limit (remember that terms $\mathcal{O}(h^2)$ and higher are discarded).

(b) Write down the geodesic equation

$$\dot{u}^\alpha + \Gamma^\alpha_{\mu\nu}u^\mu u^\nu$$

for the the metric discussed.
Show that it becomes

$$a^i + c^2\Gamma^i_{00} + 2c\Gamma^i_{0j}v^j = 0$$

if we once again discard all terms of order v^2/c^2.

(c) Show that the first term is nothing else than the expected Newtonian gravity, i.e. $-\partial_i\Phi$ (which normally is called F_i).
Show that the second term is exactly $-c\left(\vec{\nabla} \times \vec{h} \times \vec{v}\right)^i$, where $\vec{h} = (h_{01}, h_{02}, h_{03})^{\mathrm{T}}$.
Let's call $\nabla \times \vec{h} = \vec{\Omega}$.
Writing down

$$\vec{F} = m\vec{a} = m(-\vec{\nabla}\Phi + 2\vec{\Omega} \times \vec{v}),$$

is it surprising that people talk about a 'gravitomagnetic' force?

(d) Let's compare those two contributions for the effect that the sun and its rotation has on the innermost planet, Mercury.
Assume the sun is a perfect sphere of radius $R_\odot = 0.7 \times 10^9\,\mathrm{m}$ and it takes 25 days for one rotation.
Mercury is at a distance of $r = 60 \times 10^9\,\mathrm{m}$ from the sun's centre and it takes 88 days for one orbit around the sun. You can safely assume that Mercury's orbit is perpendicular to the sun's rotational axis.

2. (15 points) **Particles reacting to gravitational waves**
Let's start from the transverse-traceless gauge amplitude tensor $A_{\mu\nu}$, and consider a gravitational wave in the 1-2-plane:

$$A_{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a & b & 0 \\ 0 & b & -a & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) How does the time-dependent part of the metric $\eta_{\mu\nu} + h_{\mu\nu}(x^i, t)$ then look like if we're to describe a gravitational wave? How many components of x does h really depend on?

(b) Considering the geodesic equation from the above exercise,

$$2\Gamma^\alpha_{\mu\nu} = \eta^{\alpha\sigma}(\partial_\mu h_{\nu\sigma} + \partial_\nu h_{\mu\sigma} - \partial_\sigma h_{\mu\nu}),$$

What can we say about the spatial acceleration of a particle $\mathrm{d}^2x^i/\mathrm{d}\tau^2$ at $\tau = 0$ if we require that the particles were resting ($\dot{x}^i = 0$) at $\tau = 0$?
What are the conclusions about $x^i(\tau)$?

(c) Let's split up the line element according to

$$\mathrm{d}s^2 = c^2\mathrm{d}t^2 - \mathrm{d}l^2 - \left(\mathrm{d}x^3\right)^2,$$

then $\mathrm{d}l^2 = (\delta_{ij} - h_{ij}(t))\,\mathrm{d}x^i\mathrm{d}x^j$, where $i, j \in \{1, 2\}$.
Write h in terms of the Pauli matrices σ. What 'special' property do the ones that are needed have?

(d) Let's imagine a circle of particles P in the 1-2-plane with radius R. We can now describe the position of the particles on the circle by their angle φ.
Then we can write the *physical*[1] distance from the centre $\tilde{R}$ in phyiscal coordinates x^i_p

$$\tilde{R}^2 = (\delta_{ij} - h_{ij}(t))\,\mathrm{d}x^i_p\mathrm{d}x^j_p.$$

Define those physical coordinates on the plane (tip: they look very much like spherical ones...) and then calculate $\tilde{R}^2$ for *i*) the "+" and *ii*) the "×" modes, i.e. $a = h$, $b = 0$ and $a = 0$, $b = h$, where h is the scalar amplitude.

[1] In part (b) we've seen that x^i doesn't change, but that's due to the the metric changing! We can still define a physical distance that's indeed changing much like in cosmology *comoving* vs. *physical* distance of two galaxies!

3. (10 points) **Gravitational wave induced ellipticity**
Let's pick out our "+" mode from the above exercise, which we all know is

$$\tilde{R}^2 = R^2 \left(1 - 2h\cos(2\varphi)\cos(\omega t)\right).$$

Let's consider $h \ll 1$ (which is sensible, since we're working in weak fields)[2].

(a) Firstly, let's see how the wave looks like at maximum displacement, what needs to be fulfilled for that?

(b) Write $\tilde{R}^2$ in form of an ellipse, i.e.

$$\frac{x^2}{\epsilon_1^2} + \frac{y^2}{\epsilon_2^2} = 1.$$

What are the ϵ_i in terms of h?
tips:

- $\tilde{R}^2 = x^2 + y^2$,
- $\cos(2\varphi) = \cos^2(\varphi) - \sin^2(\varphi)$,
- only keep terms $\mathcal{O}(h)$.

(c) ϵ_1 and ϵ_2 are the semiminor and semimajor axes of an ellipse.
Calculate the eccentricity ϵ induced by a gravitational "+"-polarised wave with scalar amplitude h via

$$\epsilon = \sqrt{1 - \frac{\epsilon_2^2}{\epsilon_1^2}}.$$

4. (5 points) **Extra: Wave equations**
Why are gravitational waves dependent on the 2nd moment, i.e. quadrupole of a distribution? Why are electromagnetic waves dependent on the dipole of a distribution? What impact does this have on their spin (or polarisation)? Could you imagine a wave with spin 0?

"Von der Einführung des »λ-Gliedes« (vgl. diese Sitzungsber. 1917, S. 142) ist dabei Abstand genommen."
*–**Albert Einstein**, retracting Λ from the field equations in his 1918 publication "Über Gravitationswellen" (On Gravitational Waves)*

[2] That means that a lot of times in this exercise, you should just use the linear term. For example $(1+h)^2$ can be written as $1+2h$