# Assignment 11

Due: Week beginning 06.07.2015.

**Problem 11.1 (Critical exponents):**
Consider massive $\phi^4$-theory with mass $m$ and quartic coupling $\lambda$ in $d$ dimensions. One defines the dimensionless quantity

$$g_m = \frac{m}{\mu} \tag{1}$$

in terms of the renormalisation scale $\mu$. Following the general logic for the RG flow of dimensionful operators as discussed in the lecture, it satisfies the renormalisation group equation

$$\mu \frac{d}{d\mu} g_m = \beta_m, \qquad \beta_m = (-2 + \gamma_{\phi^2})\, g_m, \tag{2}$$

where we state without proof that $\gamma_{\phi^2} = \frac{\lambda}{16\pi^2}$ at one-loop order.

a) Show that near a renormalisation group fixed point $\lambda^*$, $g_m$ takes the form

$$g_m(\lambda(\mu)) = g_m(\lambda^*) \left(\frac{\mu}{\mu^*}\right)^{-\frac{1}{\nu}}, \tag{3}$$

where you should give the general form of $\nu$.

b) Give the mass dimension of the coupling $\lambda$ in $d$ dimensions and define the appropriate dimensionless coupling $g_\lambda$. Give its renormalisation group equation and its $\beta$-function. Use this to determine the location of a non-trivial IR fixed-point $\lambda^*$ if $d < 4$. What happens to this fixed point if $d = 4$ and if $d > 4$?

c) Argue that in a free scalar theory, the *correlation length* $\xi$ is given by $\xi = \frac{1}{m}$.
**Hint**: Show that $\langle \phi(x)\phi(0)\rangle \simeq e^{-|x|/\xi}$.
More generally, one defines the correlation length via

$$\xi = \frac{\mu^*}{p_0} \qquad \text{with} \qquad g_m(p_0) = 1. \tag{4}$$

Deduce that near a fixed point $\lambda^*$ the correlation length is given by $\xi = (g_m(\mu^*))^{-\nu}$. Evaluate $\nu$ explicitly for the above fixed-point in $d < 4$ dimensions.

d) According to Landau theory, a ferromagnet can be described by a 3-dimensional Euclidean field theory of the form

$$\mathcal{L} = \frac{1}{2}(\nabla M)^2 + b(T - T_C)M^2 + cM^4, \tag{5}$$

where $M(\vec{x})$ is the magnetisation of the ferromagnet,$T_C$ represents the so-called critical temperature and $b$, $c$ are some parameters. Argue that near $T_C$ the correlation length $\xi$

diverges.

**Hint**: Compare the system with a scalar theory and recall the form of the propagator.

Furthermore, argue that $(T - T_C)$ is the analogue of $g_m$ in the statistical field theory. Deduce the scaling law $\xi \sim (T - T_C)^{-\nu}$ near $T_C$. The exponent $\nu$ is called *critical exponent*.

**Problem 11.2 (Fermion-anti-fermion annihilation in Yang-Mills theory at tree-level):** We consider annihilation of one fermion with momentum $p_2$ and one anti-fermion with momentum $p_1$ (and both of mass $m$) into two gauge bosons with polarisation (vectors) $\epsilon^a_\mu(k_1)$ and $\epsilon^b_\nu(k_2)$ at tree-level in Yang-Mills theory. Proceed as in Figure 1. Let

$$i\mathcal{M} = i\mathcal{M}^{\mu\nu}\epsilon^{*a}_\mu(k_1)\epsilon^{*b}_\nu(k_2) \tag{6}$$

denote the amplitude, where we have explicitly factored out the dependence on the outgoing gauge boson polarisations. In the sequel we will suppress the colour index in the polarisation vectors.

a) Use the Feynman rules as stated in the lecture to compute the sum of the first two diagrams. Show that if the second outgoing gauge boson is in polarisation state $\epsilon_\mu(k_2) = (k_2)_\mu$, this expression becomes

$$i\mathcal{M}^{\mu\nu}\epsilon^*_\mu(k_1)(k_2)_\nu = (-ig)^2\bar{v}(p_1)(-i\gamma^\mu[T^a, T^b])u(p_2)\epsilon^*_\mu(k_1). \tag{7}$$

**Hint:** Use that $(\gamma \cdot p_2 - m)u(p_2) = 0 = \bar{v}(p_1)(-\gamma \cdot p_1 - m)$.

b) Use the Feynman rules to compute the third diagram and verify that again for the second outgoing gauge boson in polarisation state $\epsilon_\mu(k_2) = (k_2)_\mu$ the result is

$$i\mathcal{M}^{\mu\nu}\epsilon^*_\mu(k_1)(k_2)_\nu = g^2\bar{v}(p_1)\gamma^\mu u(p_2)\epsilon^*_\mu(k_1)f^{abc}T^c \tag{8}$$

*provided* we make the further assumption that the other gauge boson is transversely polarized, i.e. $\epsilon^\mu(k_1)(k_1)_\mu = 0$.

Show that the sum of all three diagrams cancels for the second outgoing gauge boson in polarisation state $\epsilon_\mu(k_2) = (k_2)_\mu$ and for $\epsilon^\mu(k_1)(k_1)_\mu = 0$.

**Remark**: In QED, $k^\mu M_\mu = 0$ holds without further assumptions as a consequence of the Ward identities. The above shows that the non-abelian interactions complicate things and the analogue of the Ward identities in Yang-Mills theory, the so-called Slavnov-Taylor identities, have a slightly different structure.

c) Now consider only the last diagram, which is absent in an abelian gauge theory. Evaluate it for the situation where $\epsilon(k_1) = \epsilon^+(k_1)$ and $\epsilon(k_2) = \epsilon^-(k_2)$ and show that it takes the

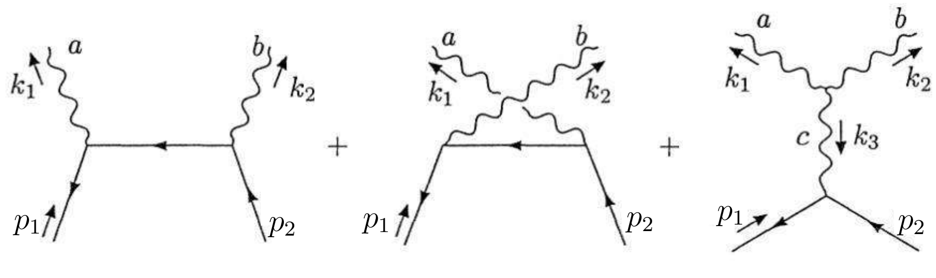


Figure 1: Diagrams contributing to fermion-anti-fermion annihilation to two gauge bosons.

non-vanishing form

$$-i\, g\, \bar{v}(p_1)\gamma_\rho\, T^c\, u(p_2) \frac{-i}{k_3^2}\, g\, f^{abc}\, k_1^\rho \frac{|\vec{k_1}|}{|\vec{k_2}|}. \tag{9}$$

Here the forward/backward light-like polarisation vectors are defined as

$$\epsilon^\pm(k) = (k^0, \pm\vec{k}). \tag{10}$$

This demonstrates that in contrast to QED, in non-abelian Yang-Mills theory non-physical null states are produced in scattering. Why is this not a problem?