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General Relativity (MKTP3) Summer Term 2015

Exercise sheet 8 2 June 2015

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1. (15 points) **Spherical symmetry and time derivatives**

   (a) Argue why the *free* Einstein field equations can also simply be written as

   $$R_{\mu\nu} = 0, \tag{1}$$

   where R is the Ricci-tensor.
   (Tip: Look at the equation you were supposed to show on sheet 7, exercise 2!)

   (b) Given the metric

   $$g_{\mu\nu} = \mathrm{diag}(B(r,t), -A(r,t), -r^2, -r^2\sin^2\theta),$$

   we all immediately see that the Christoffel symbols are as follows[1]:

   $$\begin{aligned}
   &\Gamma^0_{00} = \frac{\dot{B}}{2B}, &&\Gamma^0_{01} = \Gamma^0_{10} = \frac{B'}{2B}, &&\Gamma^1_{00} = \frac{B'}{2A},\\
   &\Gamma^0_{11} = \frac{\dot{A}}{2B}, &&\Gamma^1_{01} = \Gamma^1_{10} = \frac{\dot{A}}{2A}, &&\Gamma^1_{11} = \frac{A'}{2A},\\
   &\Gamma^1_{22} = -\frac{r}{A}, &&\Gamma^2_{12} = \Gamma^2_{21} = \frac{1}{r}, &&\Gamma^3_{13} = \Gamma^3_{31} = \frac{1}{r},\\
   &\Gamma^1_{33} = -\frac{r\sin^2\theta}{A}, &&\Gamma^2_{33} = -\sin\theta\cos\theta, &&\Gamma^3_{32} = \Gamma^3_{23} = \cot\theta.
   \end{aligned} \tag{2}$$

   In this completely unfamiliar notation, a dot $\dot{}$ denotes a t-derivative and a prime $'$ an r-derivative.
   Of the Ricci tensor,

   $$R_{\mu\nu} = \Gamma^\rho_{\mu\rho,\nu} - \Gamma^\rho_{\mu\nu,\rho} + \Gamma^\sigma_{\mu\rho}\Gamma^\rho_{\sigma\nu} - \Gamma^\sigma_{\mu\nu}\Gamma^\rho_{\sigma\rho},$$

   compute R_{10}, R_{00}, and R_{11}.

   (c) What sort of spatial symmetry is the above metric assuming?

   (d) For each individual entry of the Ricci tensor equation 1 holds.
   What does that say about $\dot{A}$ if you consider R_{10} in this vacuum (or free) case?

[1] No need to proof/show this, don't waste your ink

(e) Calculate

$$\frac{R_{00}}{B} + \frac{R_{11}}{A}.$$

(f) Using

$$\frac{R_{00}}{B} + \frac{R_{11}}{A} = 0,$$

and your previous result, show that

$$\frac{\mathrm{d}}{\mathrm{d}r} \log(AB) = 0,$$

therefore $AB = const.$

What does that say about $\dot{B}$ given your knowledge of $\dot{A}$?

If you found that $\dot{A} = \dot{B} = 0$, you just showed *Birkhoff's theorem* holds, stating that a sperically symmetric gravitational field in the absence of sources (i.e. right-hand side of Einstein field equations is 0) is static.

2. (15 points) **Piep piep kleiner Satellit**

A satellite with mass $m > 0$ is orbiting a black hole. The metric is

$$g_{\mu\nu} = \mathrm{diag}(B(r), -A(r), -r^2, -r^2 \sin^2\theta).$$

(a) Calculate

$$\frac{\mathrm{d}^2 x^0}{\mathrm{d}\lambda^2} = \frac{\mathrm{d}^2 t}{\mathrm{d}\lambda^2}$$

via the geodesic equation.

(b) Re-write the result as

$$\frac{\mathrm{d}}{\mathrm{d}\lambda}(\log f + \log g) = 0,$$

thus showing that

$$fg \equiv F = const. \tag{3}$$

What are f and g?

(c) Let's consider the equatorial plane $\theta = \pi/2$. Show that the geodesic equation for ϕ delivers something along the lines of

$$r^2 \frac{\mathrm{d}\phi}{\mathrm{d}\lambda} \equiv l = const.$$

(d) Show that the last interesting geodesic equation, i.e. the one for r, delivers us

$$\frac{\mathrm{d}^2 r}{\mathrm{d}\lambda^2} + \frac{F^2 B'}{2AB^2} + \frac{A'}{2A}\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 - \frac{l^2}{Ar^3} = 0.$$

(e) Multiplying the above result by $2A\frac{\mathrm{d}r}{\mathrm{d}\lambda}$, show that

$$A\left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 + \frac{l^2}{r^2} - \frac{F^2}{B} \equiv -\epsilon = const.$$

(f) Using

$$B(r) = A^{-1}(r) = 1 - \frac{2a}{r},$$

simplify the last result to

$$\frac{\dot{r}^2}{2} - \frac{a\epsilon}{r} + \frac{l^2}{2r^2} - \frac{al^2}{r^3} = \frac{F^2 - \epsilon}{2}. \tag{4}$$

(g) Since our satellite is not massless, we are assuming $\lambda = \tau$. Furthermore, ϵ becomes $\epsilon = c^2$ (for photons, we would have $\epsilon = 0$). Also, let's require a spherical orbit for simplicity. We know from the lecture that $a = GM/c^2$, so we can define the effective potential

$$V_{\text{eff}}(r) = -\frac{GM}{r} + \frac{l^2}{2r^2} - \frac{GMl^2}{c^2r^3}. \tag{5}$$

Which terms are old friends? Which one is new?

(h) What does equation 3 look like for our satellite? We are after the term $\mathrm{d}t/\mathrm{d}\tau$ in there, which gives us the ratio of time measured by the satellite's clock $\mathrm{d}\tau$, and an outside observer "infintely" away from the black hole $\mathrm{d}t$. Let's find this ratio! For that, plug in F from equation 4 into equation 3 for our satellite.

(i) Now we want to get rid of the l-dependency. In order to do this, let's require

$$\frac{\mathrm{d}}{\mathrm{d}r}V_{\text{eff}}(r) = 0.$$

Solve this for $l^2/(c^2r^2)$ to arrive at an expression for $\mathrm{d}t/\mathrm{d}\tau$ that only has G, M, r, and c in it!

(j) Extra: Plot V_{eff} and V'_{eff}. Is there something surprising about this?

3. (10 points) **Ricci scalar don't care**
In the lecture, you computed the four entries of the Ricci tensor $R^{\mu\nu}$, which takes on diagonal form.
Calculate

$$R = R^{\mu\nu}g_{\mu\nu}$$

and plot the result against r in units of R_S. What happens at the Schwarzschild radius? Does the Ricci scalar go crazy when it "enters" the scary black hole?

4. (5 points) **Extra: Black hole detection**
Name two ways of detecting a black hole! Since we can't see it (by definition), how would you go about looking for them in the cosmos/galaxy?

"The black holes of nature are the most perfect macroscopic objects there are in the universe: the only elements in their construction are our concepts of space and time."
*–**Subrahmanyan Chandrasekhar***