Assignment 4

Due: Week beginning 11.05.2015.

Problem 4.1 (The 1PI effective action):
In the lecture we have defined

$$\tau(x_1,\ldots,x_n) = \frac{\delta}{i\delta J(x_1)}\cdots\frac{\delta}{i\delta J(x_n)} iW[J], \tag{1}$$

$$\tilde{\Gamma}(x_1,\ldots,x_n) = \frac{\delta}{\delta\varphi(x_1)}\cdots\frac{\delta}{\delta\varphi(x_n)} i\Gamma[\varphi]. \tag{2}$$

What is the interpretation of $\tau(x_1,\ldots,x_n)$ and $\Gamma(x_1,\ldots,x_n)$?

We have presented both a computational and a graphic method to relate $\tau(x_1,x_2,x_3)$ and $\Gamma(x_1,x_2,x_3)$. Revise these methods and use them to relate $\tau(x_1,x_2,x_3,x_4)$ and $\Gamma(x_1,x_2,x_3,x_4)$. Argue that the result demonstrates that $\Gamma[\varphi]$ is indeed a generating functional of the 1PI-correlators.

Problem 4.2 (Coleman-Weinberg potential for ϕ^4-theory):
In this exercise we calculate the quantum effective action $\Gamma[\varphi]$ for the ϕ^4-theory,

$$S[\phi] = \int d^4x \left(-\tfrac{1}{2}\phi(x)(\partial^2 + m^2)\phi(x) - \tfrac{\lambda}{4!}\phi^4(x) \right), \tag{3}$$

up to one-loop order for a constant background field $\varphi = \varphi_0$. In the lecture it was shown that

$$\Gamma[\varphi] = S[\varphi] + \hbar\,\kappa^{\text{1-loop}}[\varphi] + \text{higher loop corrections} \tag{4}$$

with

$$\kappa^{\text{1-loop}}[\varphi] = -i\,\ln \int \tilde{\mathcal{D}}f\, e^{-\frac{1}{2}f\cdot\left(-\frac{i}{\hbar}\frac{\delta^2 S[\varphi]}{\delta\varphi\delta\varphi}\right)\cdot f}\,, \tag{5}$$

where the measure $\tilde{\mathcal{D}}$ was defined such that $Z[0] = 1$.

a) Show that for any Hermitian matrix A

$$\ln(\det A) = \mathrm{Tr}\,(\ln A)\ .$$

Hint: You might find it useful to do a basis change for which A becomes diagonal.

b) Evaluate $\kappa^{\text{1-loop}}$ by means of the path integral techniques presented in Assignments 2 and 3. Use, in analogy to a), that as for a differential operator D, $\ln(\det D) = \mathrm{Tr}\,(\ln D)$. How do you interpret the trace?

Hint: The result can be found in the lecture notes.

c) We define the effective potential $\Gamma[\varphi)] = -\mathrm{Vol}_{\mathbb{R}^{1,3}} V(\varphi)$. Give the effective potential up to one-loop order, $V^{\text{tree}}(\varphi = \varphi_0) + V^{1-\text{loop}}(\varphi = \varphi_0)$.

Hint: Starting from the expression found in b) you should find an integral representation of $V^{1-\text{loop}}(\varphi = \varphi_0)$ of the form

$$\int d^3k \left(\sqrt{\mathbf{k}^2 + x + \delta} - \sqrt{\mathbf{k}^2 + x}\right) . \tag{6}$$

To this end perform the integral over k^0 as a suitable contour integral. Furthermore, the following integral identity might be useful:

$$\int_x^{x+\delta} d\xi \, \frac{1}{\xi} = \ln\left(\frac{x+\delta}{x}\right) .$$

Give a physical interpretation of the two terms in (6).

Final Remark: It is not hard to show (by differentiating three-times and integrating back) that the potential up to one-loop order can be put into the form

$$\Lambda_R + \frac{1}{2} m_R^2 \varphi_0^2 + \frac{\lambda_R}{4!}\varphi_0^4 + \frac{\mu^2(\varphi_0)\,\ln \mu^2(\varphi_0)}{64\,\pi^2}, \tag{7}$$

where $\mu^2(\varphi) = m_R^2 + \frac{1}{2}\lambda_R \varphi^2$.

Here Λ_R, m_R, λ_R the suitably renormalised (and thus physical) values of the vacuum energy, mass and coupling which are made finite by absorbing all the divergences in the bare couplings. Once these are taken from experiment, what remains as a prediction is the last term, which is the world-famous Coleman-Weinberg potential term. It gives the difference between the original, classical ϕ^4-potential and the effective quantum potential, taking into account 1-loop effects.