Heidelberg University
Björn Malte Schäfer

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General Relativity (MKTP3) Summer Term 2015

Exercise sheet 4 **4 May 2015**

Tim Tugendhat (tugendhat@uni-heidelberg.de) Due: 9:15h, 11 May 2015

1. (10 points) **Straight back on the sphere**
   Last time, you derived the metric for a sphere in $\mathbb{R}^3$,
   $$g^{\mu\nu} = \mathrm{diag}(1, r^2, r^2 \sin^2(\theta)),$$
   and limited yourself to the surface. This time, we will not do so, and leave r free.
   (a) Calculate the Chrioffel symbols Γ^i_{jk} of this metric.
   (b) The geodesic equation reads
   $$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0.$$
   Do they describe straight lines? Should this depend on the coordinate system we're using?

2. (10 points) **Transforming the metric**
   If we transform the metric by a scalar factor $\mathrm{e}^{2\varphi}$, with $\varphi = \varphi(x^\mu)$ is a function of all coordinates, the metric becomes
   $$g \to g' = \mathrm{e}^{2\varphi} g \; . \tag{1}$$
   (a) By plugging in the new definition into the Christoffel symbols $\Gamma^\alpha_{\mu\nu}$, show that they will transform as
   $$\Gamma^\alpha_{\mu\nu} \to \Gamma^\alpha_{\mu\nu} + \delta^\alpha_\mu \partial_\nu \varphi + \delta^\alpha_\nu \partial_\mu \varphi - g^{\alpha\beta} g_{\mu\nu} \partial_\beta \varphi. \tag{2}$$
   (b) Null geodesics ($\mathrm{d}s = 0$), which we have seen on the last sheet, have a tangent vector that is light-like:
   $$\dot{x}^\mu \dot{x}_\mu = 0.$$
   Show that null geodesics are invariant under the transformation in a), i.e. that null geodesics of the metric g will remain null geodesics of the metric g', if the curve parameter λ is transformed as
   $$\mathrm{d}\lambda \to \mathrm{d}\lambda' = \mathrm{e}^{2\varphi} \mathrm{d}\lambda \; . \tag{3}$$

3. (20 points) **An interesting line element**
   You wander around the physics building and see the following line element written down on a blackboard:

$$ds^2 = c^2 dt^2 - a^2(t)\left[\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2(\theta)\mathrm{d}\phi^2\right)\right],$$

   and immediately think to yourself that you definitely need to know what this mysterious metric describes.

   (a) So you calculate the non-vanishing Christoffel-symbols,

   (b) and write down what geodesic a particle follows that only moves radially (i.e. in x^1-direction).

   (c) Lastly, you're interested in light moving radially in this wonderful metric (remember, $ds = 0$): You ask yourself what the radial distance to a light source,

$$r = \int_{t_\mathrm{e}}^{t_\mathrm{obs}} \mathrm{d}r',$$

   is in terms of $\mathrm{d}t$, if $k = 0$ (we'll see what this means in a future exercise).
   Next you show that, because

$$\frac{\mathrm{d}r}{\mathrm{d}t_\mathrm{e}} = 0,$$

   it must hold that

$$\frac{\mathrm{d}t_\mathrm{obs}}{\mathrm{d}t_\mathrm{e}} = \frac{a(t_\mathrm{obs})}{a(t_\mathrm{e})}.$$

   If you make the bold claim that $\mathrm{d}t = 1/f$, you can find a relation between wavelength $\lambda = c/f$ and $\frac{a(t_\mathrm{obs})}{a(t_\mathrm{e})}$. Re-write it in terms of the parameter z with

$$z = \frac{\lambda_\mathrm{obs} - \lambda_\mathrm{e}}{\lambda_\mathrm{e}}.$$

   Cosmologists might call z redshift and a the expansion factor.
   What redshift do you find for signals sent out when the Universe was only half as large as today?

4. (5 points) **Extra: Equivalence**
   If there were no space exploration, and you didn't have the means to travel very far, how could you distinguish the Earth from being a sphere that has 'normal' gravity from being flat disk accelated 'upwards' with $g = 9.81\,\mathrm{m/s^2}$ by a giant turtle?

"Tell me, why do people always say that it was natural for men to assume that the sun went around the earth rather than the earth was rotating?"
–"Well, obviously, because it just looks as if the sun is going around the earth."
–"And what would it look like if it had looked as if the earth were rotating?"
Anecdotal conversation between Ludwig Wittgenstein and a friend