Problem Sheet 3

Exercise 1: *(Representations of S_n)*

(a) Use the hook rule to determine the dimension of the irreducible representation associated to the partition

$$\lambda = (n-k, \underbrace{1, \ldots, 1}_{k\ times}, 0, \ldots, 0)\,.$$

(b) Construct the character table of S_4 by using the Frobenius formula.

Exercise 2: *(Lie groups/Lie algebras)* Recall the following notions from the lecture:

(a) What is a Lie group?
(b) What is a Lie algebra?
(c) What are left-invariant vector fields? Describe them for matrix groups.
(d) How does one obtain the structure of a Lie algebra on the tangent space of a Lie group at the identity?
(e) Show that the Lie bracket on the tangent space of the identity of matrix groups satisfies the Jacobi identity.
(f) How are the notions of homomorphisms of Lie-groups and Lie-algebras related?

Exercise 3: *($SU(2)$)* Show that $\mathrm{SU}(2) = \{A \in \mathrm{Mat}(2,2;\mathbb{C}) \mid A^\dagger A = \mathbb{I}, , \det(A) = 1\}$ can be parametrized in the following way

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \,\middle|\, a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\}.$$

Hence, $\mathrm{SU}(2)$ is homeomorphic to the three sphere $\mathbb{S}^3$, and in particular simply connected.

Exercise 4: *($SU(2)$ versus $SO(3)$)*

(a) Determine the Lie algebras of $\mathrm{SU}(2)$ and $\mathrm{SO}(3)$ and show that they are isomorphic.
(b) Given that $\mathrm{SU}(2)$ is simply connected, and $\mathrm{SO}(3)$ is connected, what does this imply for $\mathrm{SO}(3)$?
(c) Show that the center of $\mathrm{SU}(2)$ is $Z(\mathrm{SU}(2)) = \{\pm\mathbb{I}\}$. (Hint: an element Z of the center has to satisfy $A^\dagger Z A = Z$ for all $A \in \mathrm{SU}(2)$. Take a path $A(t)$ through $\mathbb{I}$ in $\mathrm{SU}(2)$ and differentiate.)
(d) Define the $\mathbb{R}$-bilinear form $< a, b >:= \mathrm{tr}(a^\dagger b)$ on the Lie algebra $T_e\mathrm{SU}(2)$. Show that as a vector space with bilinear form, $T_e\mathrm{SU}(2)$ is isomorphic to $\mathbb{R}^3$.
(e) $\mathrm{SU}(2)$ acts on $T_e\mathrm{SU}(2)$ by conjugaction $a \mapsto A^\dagger a A$ for $A \in \mathrm{SU}(2)$. Show that this action preserves the bilinear form. Hence, this defines a map $SU(2) \longrightarrow SO(3)$. What is the kernel of this map?

Please hand in solutions on June 17!