Problem Sheet 2

Exercise 1: *(Representations)* Recall the following notions from the lecture:

(a) What is a representation?
(b) What is a homomorphism (intertwiner) between two representations?
(c) What is a subrepresentation?
(d) What are reducible and irreducible representations?
(e) Recall Schur's lemma.
(f) Argue that complex representations of finite groups are fully decomposable.

Exercise 2: *(Representations of Abelian groups)*

(a) Show that all irreducible representations of a finite group G are one-dimensional if and only if G is Abelian. (Hint: What is the sum of the squares of the dimensions of the irreducible representations?)
(b) What are the irreducible representations of the cyclic group $\mathbb{Z}_n$?

Exercise 3: *(Orthogonality relation)* Let (ρ_1, V_1) and (ρ_2, V_2) be two irreducible (finite dimensional complex) representations of a finite group G. Choose basis of V_1 and V_2, and let $(\rho_1(g))_{ij}$ and $(\rho_2(g))_{ab}$ be the representation matrices with respect to these basis. Then use Schur's lemma to show:

(a) If ρ_1 and ρ_2 are inequivalent, then

$$\sum_{g\in G}(\rho_1(g^{-1}))_{ij}(\rho_2(g))_{ab} = 0$$

for all i, j, a, b.

(b) If on the other hand $\rho_1 = \rho_2$, then

$$\frac{1}{|G|}\sum_{g\in G}(\rho_1(g^{-1}))_{ij}(\rho_2(g))_{ab} = \frac{1}{\dim(V_1)}\delta_{i,b}\delta_{j,a}\,.$$

(c) Deduce that

$$\frac{1}{|G|}\sum_{g\in G}\chi_{\rho_1}(g^{-1}h)\chi_{\rho_2}(g) = \begin{cases} \frac{\chi_{\rho_1}(h)}{\dim(\rho_1)}\,, & \text{if } \rho_1 \cong \rho_2 \\ 0\,, & \text{otherwise} \end{cases}\,.$$

Exercise 4: *(Symmetric and antisymmetric products)* Let (ρ, V) be a finite dimensional representation. Consider the tensor product representation $\rho\otimes\rho$ on $V\otimes V$. Define $\sigma : V\otimes V \longrightarrow V\otimes V$ by $\sigma(v_1\otimes v_2) = v_2\otimes v_1$ for all $v_1, v_2 \in V$.

(a) Show that the tensor product decomposes into the sum $V\otimes V = S^2V \oplus \Lambda^2 V$ of spaces of symmetric and antisymmetric tensors $S^2V = \{v \in V \,|\, \sigma(v) = v\}$ and $\Lambda^2 V = \{v \in V \,|\, \sigma(v) = -v\}$, and that these are invariant subspaces in $V\otimes V$. (Hint: Consider the projectors $\frac{1}{2}(1\pm\sigma) : V\otimes V \longrightarrow V\otimes V$.)