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Heidelberg University
Björn Malte Schäfer

General Relativity (MKTP3) Summer Term 2015

Exercise sheet 1 — **13 April 2015**

Tim Tugendhat (tugendhat@uni-heidelberg.de)

Due: 9:15h, 20 April 2015

1. (10 points) **Gravitational train**

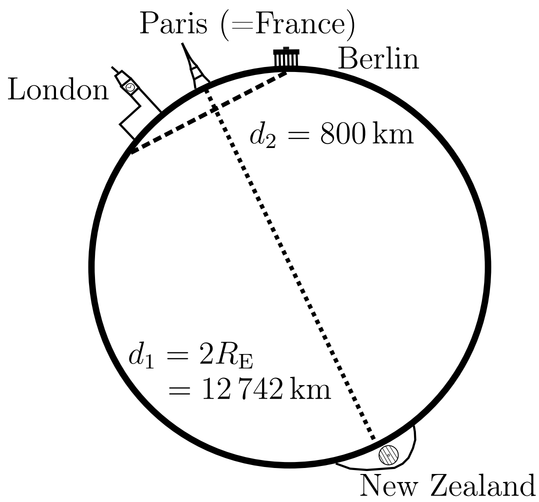

Figure 1: Map of the earth.

In order to save CO_2 emissions from air traffic, the governments of France and New Zealand have decided, very realistically, to bore a tunnel through the centre of the earth, directly connecting their landmasses with a straight tube, through which a train can travel without any friction and solely accelerated by gravitational pull.

(a) How long does it take for the train to travel one way if Earth's radial density $\rho(r)$ follows

  i. $\rho(r) = \rho_0 = 5.5\,\text{g}\,\text{cm}^{-3}$, and

  ii. $\rho(r) = \rho_1 R_\text{E}/r = 3.7\,\text{g}\,\text{cm}^{-3} R_\text{E}/r$,

  for $0 < r \leq R_\text{E}$.

(b) Assume there was a similar tunnel between Berlin and London. How long would that journey take, assuming constant density ρ_0?

(c) What is the period T of a hypothetical satellite that orbits the Earth on its surface? Why is it exactly the same as the time for the return journey for the train?

2. (10 points) **Lagrangian uniqueness**
Consider the Langrangian for point mass within a potential V,

$$\mathcal{L}(q, \dot{q}, t) = \frac{1}{2} m\dot{q}^2 - V(q), \tag{1}$$

and

(a) show that the equations of motion (Euler-Lagrange equations) won't change for transformations of the type

$$\mathcal{L} \to \mathcal{L}'(q, \dot{q}, t) = \alpha \mathcal{L}(q, \dot{q}, t) + \frac{\mathrm{d}}{\mathrm{d}t} f(q, t), \tag{2}$$

or, in other words, that the action

$$S = \int \mathrm{d}t \, \mathcal{L}(q, \dot{q}, t)$$

is invariant under scaling by an arbitrary number α and gauge transformations which add a total time derivative of an arbitrary function $f(q, t)$ to the Lagrangian.

(b) What is the physical interpretation of the property you just showed?

(c) Show that under the infinitesimal transformation $q \to q + \mathrm{d}q$, the Lagrangian gains a contribution of potential energy that can be written in the familiar form $F\mathrm{d}s$. This means that in this case, we don't have homogeneity of space.

(d) Give an example of a symmetry that the above Lagrangian $\mathcal{L}$ does exhibit, and name the associated conserved quantity.

3. (10 points) **Newton's space rope**
   Consider a rope that is connected to the surface of the Earth on the equator. It is affected by the gravitational force

$$F_{\mathrm{G}}(r) = G\frac{Mm}{r^2},$$

   as well as a centrifugal force

$$F_{\mathrm{C}}(r) = \frac{mv^2}{r} = m\omega^2 r.$$

   (a) Write the forces as integrals along the rope axis. Assume a constant length-density σ.
   (b) At what point do the two forces cancel out, i.e. how long does the rope have to be to be in an equilibrium?
   Use the dimensionless variable $\xi = x/R_{\mathrm{E}}$ to express its length in Earth radii.
   (c) What is the orbital velocity of the outermost point of the rope? (You can assume that the rope is rigid)

4. (10 points) **Keplerian orbits**
   Given the Lagrangian

$$\mathcal{L} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) - U(r), \tag{3}$$

   we're going to find whether Keplerian orbits are closed.

   (a) Name q_i and $\dot{q}_i$. What assumption about the shape of the orbits has been made? Is this a sensible choice given what you know about planetary orbits?
   (b) What is the physical interpretation of the three summands?
   (c) What is the total Energy of the system $\mathcal{E} = \sum E_i$? Where applicable, write it in terms of the canonical momentum in φ-direction,

$$p_\varphi = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}.$$

   (d) From the above, you can easily arrive at

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \sqrt{\frac{2}{m}\left(\mathcal{E} - U(r)\right) - \frac{p_\varphi^2}{m^2 r^2}}. \tag{4}$$

Try to express $\mathrm{d}\varphi$ in terms of $\mathrm{d}r$! (Tip: $p_\varphi = mr^2\frac{\mathrm{d}\varphi}{\mathrm{d}t}$).
Extra question: What condition needs to apply that φ describes a closed curve?

(e) From equation 4, we can immediately see that $\dot{r} = 0$ iff

$$\mathcal{E} = U + \frac{p_\varphi^2}{2m^2r^2} = U + \frac{1}{2}mr^2\dot{\varphi}^2.$$

What very special form of orbit do we have in this case of $\dot{r} = 0$? Given

$$U = G\frac{Mm}{r},$$

produce a plot of $\mathcal{E}(r)$ for
$M = 2 \times 10^{30}\,\mathrm{kg}$,
$m = 1 \times 10^{24}\,\mathrm{kg}$,
$\dot{\varphi} = 2 \times 10^{-7}\,\mathrm{s}^{-1}$.
If done correctly, you should see a minimum at $r \approx 150 \times 10^6\,\mathrm{km}$. Guess whose potential you just plotted!

5. (5 points) **Extra: Mechanical similarty**

(a) A team of astronauts lands on Mars. They transmit their first steps on the red planet live on TV and in order to prove that it's not a hoax, one of them uses a simple pendulum of length $l = 1\,\mathrm{m}$ to show to the audience that they are really on Mars.
As the gravitational constant g is supposedly lower on Mars than in a TV studio on Earth, the pundulum should have a larger period $T \propto \sqrt{l/g}$.
Can you, as TV audience, distinguish between a lower gravitational constant g and a slower passage of time?
Could this still be a hoax?

"There is no royal road to geometry."
– *Euclid*