Problem Sheet 1

Exercise 1: *(Definitions)*

(a) Define what a group is and show from the axioms that left-identity and -inverses are also right-identity and -inverses, and that identity and inverses are unique.
(b) What is a group homomorphism? Show that the map $\mathbb{Z} \longrightarrow \mathbb{Z}_n$, $z \longmapsto [z]$ is a group homomorphism.
(c) What is a subgroup? Show that the rotations form a subgroup $\mathbb{Z}_n \subset D_n$ of the dihedral group D_n.
(d) What are conjugacy classes? Determine all conjugacy classes of D_4 and D_5. What about general D_n?
(e) What is a normal subgroup $H \triangleleft G$ of a group G? Give an example of a non-trivial normal subgroup of D_5. Show that kernels $\ker(\varphi) = \{g \in G \,|\, \varphi(g) = e\}$ of group homomorphisms $\varphi \in \mathrm{Hom}(G, G')$ are normal subgroups of G. Argue that the special unitary $n \times n$-matrices $\mathrm{SU}(n) = \{A \in \mathrm{U}(n) \,|\, \det(A) = 1\}$ form a subgroup of $\mathrm{U}(n)$ and that it is normal: $\mathrm{SU}(n) \triangleleft \mathrm{U}(n)$.
(f) Show that for any normal subgroup $H \triangleleft G$, the quotient G/H has the structure of a group. Show that $\mathbb{Z}_n \triangleleft D_n$ and $D_n/\mathbb{Z}_n \cong \mathbb{Z}_2$.
(g) Let N and H be groups and $\vartheta \in \mathrm{Hom}(H, \mathrm{Aut}(N))$ be a group homomorphism from the group H into the group of invertible group homomorphisms from N to itself. Show that the product set $N \times H$ together with the multiplication

$$(n, h) \cdot (n', h') = (n\vartheta(h)(n'), hh')$$

defines a group. It is called the semidirect product and denoted by $N \rtimes_\vartheta H$.

Exercise 2: *(Finite groups)*

(a) Let G be a finite group with a prime number of elements. What can you say about G?
(b) Write the multiplication table of $\mathbb{Z}_3$ and the respective embedding $\mathbb{Z}_3 \hookrightarrow S_3$ into the symmetric group.

Exercise 3: *(Symmetric group)*

(a) Write the permutation

$$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 1 & 5 & 2 & 4 & 6 \end{pmatrix}$$

as product of non-intersecting cycles.
(b) What is the order of S_3? How many conjugacy classes does S_3 have? Determine their sizes, and give a representative of each one.
(c) Show that $H = \{e, (1, 2)\} \subset S_3$ is a subgroup. Is it normal?