Quantum Field Theory I
Lecturer: Jürgen Berges
Tutor: Viraf Mehta

Institut für Theoretische Physik
Universität Heidelberg
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Mid-semester Exam on Quantum Field Theory I

Week beginning: 8th December, 2014

Duration: 1.5 hours

- On each sheet of paper you hand in, you must write your name, matriculation number and group number.
- Every solution to a problem should start on a new sheet.
- This mock exam will not count toward your final grade. However, it will give an indication of your current progress and understanding of the material covered so far.

Good Luck!

Obtained Points:

Problem:	1.	2.	$\sum$
Points:	/5	/5	/10

Practise group

Matriculation Number

Name

Problem 1 (5 Points)**:**
Consider a free complex scalar field $\Phi(x)$ with action

$$S = \int d^4x \Big(\partial^\mu \Phi^\dagger(x)\, \partial_\mu \Phi(x) - m^2\, \Phi^\dagger(x)\Phi(x) \Big). \tag{1}$$

a.) Derive the equation of motion for $\Phi(x)$. / Pt.

b.) Derive the expression for the canonically conjugate momentum density to $\Phi(x)$ and compute the Hamiltonian density $\mathcal{H}$. / Pt.

c.) The mode expansion for $\Phi(x)$ takes the form / Pt.

$$\Phi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \Big(a(\vec{p}) e^{-ip\cdot x} + b^\dagger(\vec{p}) e^{ip\cdot x} \Big). \tag{2}$$

Make an Ansatz for the commutation relations of the modes $a(\vec{p})$, $a^\dagger(\vec{p})$, $b(\vec{p})$ and $b^\dagger(\vec{p})$ such that when you explicitly compute the equal time commutator $[\Phi(t,\vec{x}), \dot{\Phi}^\dagger(t,\vec{y})]$ you obtain the right result for canonical quantisation.

d.) Show that the action (1) is invariant under the transformation / Pt.

$$\Phi(x) \to e^{i\alpha}\Phi(x), \qquad \alpha \in \mathbb{R} \tag{3}$$

and derive the associated Noether current and the Noether charge Q in terms of the fundamental fields.
Hint: $Q = \int d^3x\; j^0(x)$ and $j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi - F^\mu$.

e.) The expression for the Noether charge Q in terms of the modes reads (in some overall normalisation) / Pt.

$$Q = \int \frac{d^3p}{(2\pi)^3} \Big(a^\dagger(\vec{p}) a(\vec{p}) - b^\dagger(\vec{p}) b(\vec{p}) \Big). \tag{4}$$

Compute the charge, i.e. the eigenvalue with respect to Q, of the states $a^\dagger(\vec{p})|0\rangle$ and $b^\dagger(\vec{p})|0\rangle$.

Problem 2 (5 points)**:**
This time consider the real scalar field $\phi(x)$.

a.) Show that the time-ordered product $\mathrm{T}\left(\phi(x_1)\phi(x_2)\right)$ and the normal-ordered product $:(\phi(x_1)\phi(x_2)):$ are both symmetric under the interchange of x_1 and x_2. $/\frac{1}{2}$ Pt.

b.) Deduce that the Feynman propagator, $D_F\left(x_1 - x_2\right)$, has the same symmetry property. $/\frac{1}{2}$ Pt.

c.) Check Wick's theorem for the case of three scalar fields, / Pt.

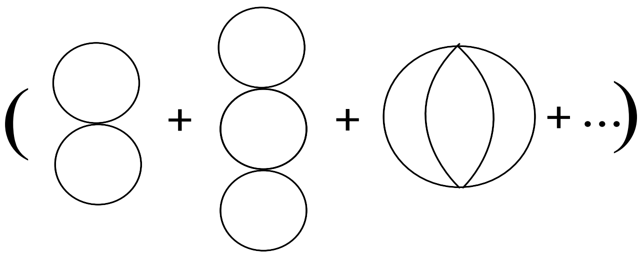
$$\begin{aligned}\mathrm{T}\left(\phi\left(x_1\right)\phi\left(x_2\right)\phi\left(x_3\right)\right) = &:\left(\phi\left(x_1\right)\phi\left(x_2\right)\phi\left(x_3\right)\right): + \phi(x_1)D_F\left(x_2 - x_3\right) \\ &+ \phi(x_2)D_F\left(x_3 - x_1\right) + \phi(x_3)D_F\left(x_1 - x_2\right).\end{aligned} \tag{5}$$

Now consider an interacting real scalar field theory with Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4. \tag{6}$$

d.) Examine $\langle 0|S|0\rangle$ to order λ^2 in this theory and identify the different diagrams that arise from an application of Wick's theorem. / Pt.
Hint: $\langle 0|S|0\rangle = \langle 0|T\exp\left[-i\int_{-\infty}^{\infty} dt\, H_I(t)\right]|0\rangle$.

e.) Confirm that, to order λ^2, the combinatoric factors work out such that the vacuum-to-vacuum amplitude is given by the exponential sum of distinct vacuum bubble types, /2 Pt.

$\langle 0|S|0\rangle = \exp$ 

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