27.04.2015

# Quantum Field Theory II - Assignment 2

## Problem 2.1 (Path integral and time-ordering)

Show, by using the steps of the derivation of the path integral presented in the lecture, that

$$\int_{q(t_I)=q_I}^{q(t_F)=q_F} \mathcal{D}q(t)\,\mathcal{D}p(t)\; q(t_1)\,q(t_2)\, e^{iS[p,q]} = \langle q_F, t_F | T\, q_H(t_1)\, q_H(t_2) | q_I, t_I\rangle. \qquad (1)$$

Put special emphasis on showing how the time-ordering appears.

$$\int_{q(t_I)=q_I}^{q(t_F)=q_F} \mathcal{D}q(t)\,\mathcal{D}p(t)\; q(t_1)\,q(t_2)\, e^{iS[p,q]} := \int_{q(t_I)=q_I}^{q(t_F)=q_F} \mathcal{D}q(t)\,\mathcal{D}p(t)\; q(t_1)\,q(t_2)\, e^{i\int_{t_I}^{t_F} dt\, L(p,q)}$$

$$= \int_{q(t_I)=q_I}^{q(t_F)=q_F} \mathcal{D}q(t)\,\mathcal{D}p(t)\; q(t_1)\,q(t_2)\, e^{i\int_{t_I}^{t_F} dt\,(p\dot{q} - H(p,q))} = \lim_{N\to\infty} \int \frac{dp_0}{2\pi} \prod_{k=1}^{N} \frac{dp_k\, dq_k}{2\pi}\, q(t_1)\, q(t_2)\, e^{i\sum_{k=0}^{N}\left[p_k \frac{q_{k+1}-q_k}{\delta t} - H(p_k,\bar{q}_k)\right]\delta t}$$

where $q_0 \equiv q_I$ and $q_{N+1} \equiv q_F$. The derivative of $q$ became $\dot{q} \to \frac{q_{k+1}-q_k}{\delta t}$ for $\delta t > 0$ discrete.

$$= \lim_{N\to\infty} \int \frac{dp_0}{2\pi} \prod_{k=1}^{N} \frac{dp_k\, dq_k}{2\pi}\, q(t_1)\, q(t_2)\, e^{i\sum_{k=0}^{N} p_k q_{k+1}}\, e^{-i\sum_{k=0}^{N} p_k q_k}\, e^{-i\sum_{k=0}^{N} H(p_k, \frac{1}{2}(q_k+q_{k+1}))\,\delta t}.$$

We inserted $\bar{q}_k = \frac{1}{2}(q_k + q_{k+1})$ when discretizing the time $t$ and now require $H(p,q)$ to decompose such that $H(p,q) = f(p) + V(q)$.

$$= \lim_{N\to\infty} \int dp_0 \prod_{k=1}^{N} dp_k\, dq_k\; q(t_1)\, q(t_2) \prod_{k=0}^{N} \langle q_{k+1}|p_k\rangle \langle p_k | q_k\rangle\, e^{-i\sum_{k=0}^{N} f(p_k)\,\delta t}\, e^{-i\sum_{k=0}^{N} V(q)\,\delta t},$$

where we used $\frac{1}{\sqrt{2\pi}} e^{ip_k q_{k+1}} = \langle q_{k+1}|p_k\rangle$ and $\frac{1}{\sqrt{2\pi}} e^{-ip_k q_k} = \langle p_k | q_k\rangle$.

$$= \lim_{N\to\infty} \int dp_0 \prod_{k=1}^{N} dp_k\, q_k\; q(t_1)\, q(t_2) \prod_{k=0}^{N} \langle q_{k+1}| e^{-if(p_k)\delta t} |p_k\rangle \langle p_k | e^{-iV(q_k)\delta t} | q_k\rangle$$

$$= \lim_{N\to\infty} \int dp_0 \prod_{k=1}^{N} dp_k\, dq_k\; q(t_1)\, q(t_2) \prod_{k=0}^{N} \langle q_{k+1}| e^{-if(\hat{p})\delta t} |p_k\rangle \langle p_k | e^{-iV(\hat{q})\delta t} | q_k\rangle$$

$$= \lim_{N\to\infty} \int \prod_{k=1}^{N} dq_k\; q(t_1)\, q(t_2) \prod_{k=0}^{N} \langle q_{k+1}| e^{-if(\hat{p})\delta t}\, e^{-iV(\hat{q})\delta t} | q_k\rangle.$$

Here, we insert a zero in the form of $e^{\frac{1}{2}[f(\hat{p}), V(\hat{q})]\delta t^2 + \mathcal{O}(\delta t^3)}$, which vanishes when performing the limit $\lim_{N\to\infty}$. We can apply the Baker-Campbell-Hausdorff formula backwards to get

$$\int_{q(t_I)=q_I}^{q(t_F)=q_F} \mathcal{D}q(t)\,\mathcal{D}p(t)\, q(t_1)\, q(t_2)\, e^{iS[p,q]} = \lim_{N\to\infty}\int \prod_{k=1}^{N} dq_k\, q(t_1)\, q(t_2) \prod_{k=0}^{N} \langle q_{k+1}| e^{-iT(\hat p)\delta t} e^{-iV(\hat q)\delta t} e^{\frac{1}{2}[T(\hat p),V(\hat q)]\delta t^2 + \dots} |q_k\rangle$$

$$= \lim_{N\to\infty}\int \prod_{k=1}^{N} dq_k\, q(t_1)\, q(t_2) \prod_{k=0}^{N} \langle q_{k+1}| e^{-i\hat H\delta t} |q_k\rangle.$$

Now follows the part, where time-ordering is introduced. Since $q(t_1)$ and $q(t_2)$ are still simple fcts, they can be commuted freely throughout the integral. The trick is to move them into the right spot within the product of $q$-eigenstates such that they act on the correct eigenstate to form an eigenvalue eq.,

$$= \lim_{N\to\infty}\int \prod_{k=1}^{N} dq_k \begin{cases} \prod_{k=j+1}^{N}\langle q_{k+1}| e^{-i\hat H\delta t}|q_k\rangle \langle q_{j+1}| e^{-i\hat H\delta t} \underbrace{q(t_1)|q_j\rangle|_{t_1}}_{\hat q_S|q_j\rangle|_{t_1}} \prod_{k=i+1}^{j-1}\langle q_{k+1}| e^{-i\hat H\delta t}|q_k\rangle\langle q_{i+1}| e^{-i\hat H\delta t} \underbrace{q(t_2)|q_i\rangle|_{t_2}}_{\hat q_S|q_i\rangle|_{t_2}} \prod_{k=0}^{i-1}\langle q_{k+1}| e^{-i\hat H\delta t}|q_k\rangle & \text{for } t_1>t_2 \\ \prod_{k=j+1}^{N}\langle q_{k+1}| e^{-i\hat H\delta t}|q_k\rangle \langle q_{j+1}| e^{-i\hat H\delta t} \underbrace{q(t_2)|q_j\rangle|_{t_2}}_{\hat q_S|q_j\rangle|_{t_2}} \prod_{k=i+1}^{j-1}\langle q_{k+1}| e^{-i\hat H\delta t}|q_k\rangle\langle q_{i+1}| e^{-i\hat H\delta t} \underbrace{q(t_1)|q_i\rangle|_{t_1}}_{\hat q_S|q_i\rangle|_{t_1}} \prod_{k=0}^{i-1}\langle q_{k+1}| e^{-i\hat H\delta t}|q_k\rangle & \text{for } t_2>t_1 \end{cases}$$

$$= \begin{cases} \langle q_F| e^{-i\hat H(t_F-t_1)}\,\hat q_S\, e^{-i\hat H(t_1-t_2)}\,\hat q_S\, e^{-i\hat H(t_2-t_I)}|q_I\rangle & \text{for } t_1>t_2 \\ \langle q_F| e^{-i\hat H(t_F-t_2)}\,\hat q_S\, e^{-i\hat H(t_2-t_1)}\,\hat q_S\, e^{-i\hat H(t_1-t_I)}|q_I\rangle & \text{for } t_2>t_1 \end{cases}$$

$$= \begin{cases} \langle q_F| e^{-i\hat H t_F}\,\hat q_H(t_1)\,\hat q_H(t_2)\, e^{i\hat H t_I}|q_I\rangle & \text{for } t_1>t_2 \\ \langle q_F| e^{-i\hat H t_F}\,\hat q_H(t_2)\,\hat q_H(t_1)\, e^{i\hat H t_I}|q_I\rangle & \text{for } t_2>t_1 \end{cases}$$

$$= \left\{ \begin{array}{ll} \langle q_F, t_F|\,\hat q_H(t_1)\,\hat q_H(t_2)\,|q_I, t_I\rangle & \text{for } t_1>t_2 \\ \langle q_F, t_F|\,\hat q_H(t_2)\,\hat q_H(t_1)\,|q_I, t_I\rangle & \text{for } t_2>t_1 \end{array} \right\} = \langle q_F, t_F|\,T\hat q_H(t_1)\,\hat q_H(t_2)\,|q_I, t_I\rangle$$

Above, in the line where we performed all the $q$-integrals, $\int dq_k\, |q_k\rangle\langle q_k| = 1$, we we used that $\delta t = \frac{T}{N+1} = \frac{t_F - t_I}{N+1}$, from which it follows, that $(N-j)\,\delta t = t_F - t_1$, $((j-1)-i)\,\delta t = t_1 - t_2$, etc.

We now move on to a shorter and more elegant proof of eq. (1) that is also more intrinsic to the path integral formulation. It works via dissection of the path integral. We operate here under the assumption that $t_F > t_1 > t_2 > t_I$ or $t_F > t_2 > t_1 > t_I$, i.e. that $t_1$ is truly larger / smaller than $t_2$. If $t_1 = t_2$, time-ordering is obsolete and there is nothing to show.

$$\int_{q(t_I)=q_I}^{q(t_F)=q_F} \mathcal{D}q(t)\,\mathcal{D}p(t)\, q(t_1)\, q(t_2)\, e^{iS[p,q]} := \lim_{N\to\infty} \int \frac{dp_0}{2\pi} \prod_{k=1}^{N} \frac{dp_k\, dq_k}{2\pi}\, q(t_1)\, q(t_2)\, e^{i\sum_{k=0}^{N} \underbrace{\left(p_k \frac{q_{k+1}-q_k}{\delta t} - H(p_k, \bar{q}_k)\right)\delta t}_{= L(p_k, \bar{q}_k)}}$$

The product $\prod_{k=1}^{N}$ we now split up at $j$ such that $j\,\delta t = \frac{1}{2}|t_1 - t_2|$. This ensures that $q(t_1)$ and $q(t_2)$ can each be evaluated solely in one of the two resulting integrals and that they can be evaluated separately from one another. To actually perform the path integral split up, we may trouble Fubini's theorem and refer to the well-definedness of the path integral in quantum mechanics as demonstrated in section 7.1 of the script. To save on writing, we hence assume $t_1 > t_2$.

$$= \lim_{N\to\infty} \int \frac{dp_0}{2\pi} \prod_{k=1}^{j-1} \frac{dp_k\, dq_k}{2\pi}\, q(t_1)\, e^{i\sum_{k=0}^{N} L(p_k, \bar{q}_k)} \int dq_j \int \frac{dp_j}{2\pi} \prod_{k=j+1}^{N} \frac{dp_k\, dq_k}{2\pi}\, q(t_2)\, e^{i\sum_{k=0}^{N} L(p_k, \bar{q}_k)}$$

In the above line, we note that our split up created two new path integrals and an integral over $q_{j+1}$ in between. The expression can therefore be rewritten, again using Fubini's theorem to rearrange integrals, as

$$= \int dq_{j+1} \int_{q(t_I)=q_I}^{q(t_j)=q_j} \mathcal{D}q(t)\,\mathcal{D}p(t)\, q(t_1)\, e^{iS[p,q]} \int_{q(t_j)=q_j}^{q(t_f)=q_f} \mathcal{D}q(t)\,\mathcal{D}p(t)\, q(t_2)\, e^{iS[p,q]}$$

$$= \int dq_j \; \langle q_F, t_F | \hat{q}_H(t_1) | q_j, t_j \rangle \langle q_j, t_j | \hat{q}_H(t_2) | q_I, t_I \rangle$$

$$= \int dq_j \, \langle q_F, t_F | \hat{q}_H(t_1)\, e^{i\hat{H}t_j} | q_j \rangle \langle q_j | e^{-i\hat{H}t_j} \hat{q}_H(t_2) | q_I, t_I \rangle = \langle q_F, t_F | \hat{q}_H(t_1)\, e^{i\hat{H}t_j} e^{-i\hat{H}t_j}\, \hat{q}_H(t_2) | q_I, t_I \rangle$$

$$= \langle q_F, t_F | \hat{q}_H(t_1)\, \hat{q}_H(t_2) | q_I, t_I \rangle = \langle q_F, t_F | T \hat{q}_H(t_1)\, \hat{q}_H(t_2) | q_I, t_I \rangle,$$

where in the last step, time-ordering could be added trivially since we set $t_1 > t_2$. However, it is important to note that we could equally well have set

$t_2 > t_1$, in which case following the same procedure as we did just now, $q(t_2)$ would have ended up in the left path integral and $q(t_1)$ in the right one, reversing their order in the final expression. This goes to show that the path integral formulation of quantum field theory takes care of time-ordering automatically.

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## Problem 2.2 (Quantum mechanical oscillator and path integral)

The transition amplitude between a state $|q_a\rangle$ at $t=0$ and $|q_b\rangle$ at $t=T$ can be schematically expressed as

$$\langle q_b| e^{-iHt} |q_a\rangle = \int_{q(0)=q_a}^{q(T)=q_b} \mathcal{D}q \, e^{iS[q]} \tag{2}$$

where the integral $\int \mathcal{D}q$ is performed over all the possible trajectories connecting the points $q(0)=q_a$ and $q(T)=q_b$, and $S[q]=\int_0^T dt\, \mathcal{L}(q,\dot q)$.
In the case of free fields, $H=\frac{\hat p^2}{2m}$, the transition amplitude is simple to calculate. Show that

$$\langle q_b| e^{-i\frac{\hat p^2}{2m}T} |q_a\rangle = \sqrt{\frac{m}{2\pi i T}}\, e^{i\frac{m}{2T}(q_b-q_a)^2}$$

$$\langle q_b| e^{-i\frac{\hat p^2}{2m}T} |q_a\rangle = \langle q_b| e^{-i\frac{\hat p^2}{2m}T} \underbrace{\int dp\, |p\rangle\langle p|}_{\mathbb{1},\text{ by the completeness relation}} q_a\rangle = \int dp\, \langle q_b| e^{-i\frac{\hat p^2}{2m}T} |p\rangle\langle p|q_a\rangle$$

$$= \int dp\, \langle q_b| e^{-i\frac{p^2}{2m}T} |p\rangle\langle p|q_a\rangle = \int dp\, \underbrace{\langle q_b|p\rangle}_{\frac{1}{\sqrt{2\pi}}e^{ipq_b}} \underbrace{\langle p|q_a\rangle}_{\frac{1}{\sqrt{2\pi}}e^{-ipq_a}} e^{-i\frac{p^2}{2m}T} = \int \frac{dp}{2\pi} e^{-ip(q_b-q_a)} e^{-i\frac{p^2}{2m}T}$$

$$= \int \frac{dp}{2\pi} \exp\left[-\frac{iT}{2m}\left(p+\frac{m}{T}(q_b-q_a)\right)^2 + i\frac{m}{2T}(q_b-q_a)^2\right], \quad \text{Gauss integral: } \int e^{-a(x-b)^2}dx = \sqrt{\frac{\pi}{a}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{\pi}{\frac{iT}{2m}}}\, e^{i\frac{m}{2T}(q_b-q_a)^2} = \sqrt{\frac{m}{2\pi i T}}\, e^{i\frac{m}{2T}(q_b-q_a)^2}$$

We are now going to determine in a few steps the transition amplitude of an harmonic oscillator. The system is characterized by the Lagrangian density

$$\mathcal{L} = \frac{m}{2}\dot q^2 - \frac{m\omega^2}{2} q^2$$

a) Recall that the classical trajectory $q_c(t)$ is found by extremising the action,

$$\frac{\delta S[q(t)]}{\delta q(t)} = -m\ddot q(t) - m\omega^2 q(t) = 0,$$

and imposing the boundary conditions $q(0)=q_a$ and $q(T)=q_b$. An arbitrary trajectory $q(t)$ can then be decomposed as $q(t) = q_c(t) + y(t)$ with the boundary conditions $y(0) = y(T) = 0$.

$$\frac{\delta S[q(t)]}{\delta q(t)} = \frac{\delta}{\delta q(t)} \int_0^T dt' \mathcal{L} = \frac{\delta}{\delta q(t)} \int_0^T dt' \left( \frac{m}{2} \dot{q}^2(q,t') - \frac{m}{2} \omega^2 q^2(t') \right) = \int_0^T dt' \left( \frac{m}{2} \frac{\delta \dot{q}^2(q,t')}{\delta q(t)} - \frac{m}{2} \omega^2 \frac{\delta q^2(t')}{\delta q(t)} \right)$$

$$= \int_0^T dt' \left( m \dot{q}(q,t') \frac{\delta \dot{q}(q,t')}{\delta q(t)} - m\omega^2 q(t') \underbrace{\frac{\delta q(t')}{\delta q(t)}}_{\delta(t-t')} \right) = \int_0^T dt' \left( m \dot{q}(q,t') \frac{d}{dt'} \underbrace{\frac{\delta q(t')}{\delta q(t)}}_{\delta(t-t')} - m\omega^2 q(t') \delta(t-t') \right)$$

$$= \underbrace{m \dot{q}(q,t') \frac{\delta q(t')}{\delta q(t)} \Big|_0^T}_{0} + \int_0^T dt' \left( -m \frac{d}{dt'} \dot{q}(q,t') - m\omega^2 q(t') \right) \delta(t-t')$$

$$= -m \ddot{q}(q,t) - m\omega^2 q(t) \stackrel{!}{=} 0 \implies m\ddot{q} + m\omega^2 q = 0$$

The boundary terms coming from the partial integration vanish, because the boundary points $q(t=0) = q_a$ and $q(t=T) = q_b$ are held fixed, meaning their variation vanishes, i.e.

$$\frac{\delta q(t'=T)}{\delta q(t)} = \frac{\delta q(t'=0)}{\delta q(t)} = 0.$$

b) Convince yourself that the following expression for the action is exact

$$S[q] = S[q_c] + \int_0^T dt \left. \frac{\delta S[q]}{\delta q(t)} \right|_{q(t)=q_c(t)} y(t) + \frac{1}{2} \int_0^T \int_0^T dt\, dt' \left. \frac{\delta^2 S[q]}{\delta q(t) \delta q(t')} \right|_{q(t)=q_c(t)} y(t)\, y(t'). \quad (4)$$

Then show that we can write

$$\frac{\delta^2 S[q]}{\delta q(t) \delta q(t')} = -m \frac{d^2}{dt^2} \delta(t-t') - m\omega^2 \delta(t-t') \quad (5)$$

so that

$$S[q] = S[q_c] + \frac{m}{2} \int_0^T dt \left( \dot{y}^2(t) - \omega^2 y^2(t) \right) =: S[q_c] + S[y]. \quad (6)$$

Then our initial amplitude reads

$$\int \mathcal{D}q\, e^{iS[q]} = e^{iS[q_c]} \int \mathcal{D}y\, e^{iS[y]}. \quad (7)$$

To do the convincing, we choose to first show eq. (5). In part a), we already showed that $\frac{\delta S[q]}{\delta q(t)} = -m\ddot{q}(t) - m\omega^2 q(t)$. Therefore

$$\frac{\delta^2 S[q]}{\delta q(t)\,\delta q(t')} = \frac{\delta}{\delta q(t)}\frac{\delta S[q]}{\delta q(t')} = \frac{\delta}{\delta q(t)}\left(-m\ddot{q}(t') - m\omega^2 q(t')\right) = -m\frac{\delta \ddot{q}(t')}{\delta q(t)} - m\omega^2\frac{\delta q(t')}{\delta q(t)}$$

$$= -m\frac{d^2}{d^2t}\frac{\delta q(t')}{\delta q(t)} - m\omega^2\frac{\delta q(t')}{\delta q(t)} = -m\frac{d^2}{d^2t}\delta(t-t') - m\omega^2\delta(t-t').$$

Using this result, we can make sure eq. (4) is correct

$$S[q] = \int_0^T dt\, \mathcal{L}(q(t),\dot{q}(t),t) = \int_0^T dt\left(\frac{m}{2}\left(\dot{q}_c(t)+\dot{y}(t)\right)^2 - \frac{m}{2}\omega^2\left(q_c(t)+y(t)\right)^2\right)$$

$$= \underbrace{\int_0^T dt\left(\frac{m}{2}\dot{q}_c^2(t) - \frac{m}{2}\omega^2 q_c^2(t)\right)}_{S[q_c]} + \int_0^T dt\left(m\dot{q}_c(t)\dot{y}(t) - m\omega^2 q_c(t)y(t)\right) + \int_0^T dt\left(\frac{m}{2}\dot{y}^2(t) - \frac{m}{2}\omega^2 y^2(t)\right)$$

$$= S[q_c] + \underbrace{m\dot{q}_c(t)y(t)\Big|_0^T}_{0,\text{ since } y(0)=y(T)=0} + \int_0^T dt\,\underbrace{\left(-m\ddot{q}_c(t) - m\omega^2 q_c(t)\right)}_{\frac{\delta S[q]}{\delta q(t)}} y(t)$$

$$+ \frac{1}{2}\int_0^T dt\int_0^T dt'\left(m\dot{y}(t)\dot{y}(t')\delta(t-t') - m\omega^2 y(t)y(t')\delta(t-t')\right)$$

$$= S[q_c] + \int_0^T dt\,\frac{\delta S[q]}{\delta q(t)}y(t) + \underbrace{\frac{1}{2}m y(t)\int_0^T dt'\,\dot{y}(t')\delta(t-t')\Big|_0^T}_{0} + \frac{1}{2}\int_0^T dt\int_0^T dt'\left(-m y(t)\dot{y}(t')\frac{d}{dt}\delta(t-t') - \dots\right)$$

$$= S[q_c] + \int_0^T dt\,\frac{\delta S[q]}{\delta q(t)}y(t) - \underbrace{\frac{1}{2}m y(t)\int_0^T dt\,y(t)\frac{d}{dt}\delta(t-t')\Big|_0^T}_{0}$$

$$+ \frac{1}{2}\int_0^T dt\int_0^T dt'\left(m y(t)y(t')\frac{d^2}{d^2t}\delta(t-t') - m\omega^2 y(t)y(t')\delta(t-t')\right)$$

$$= S[q_c] + \int_0^T dt\,\frac{\delta S[q]}{\delta q(t)}y(t) + \frac{1}{2}\int_0^T dt\int_0^T dt'\,\frac{\delta^2 S[q]}{\delta q(t)\,\delta q(t')}y(t)y(t')$$

A simpler and more general way of proving the exactness of eq. (4) is acknowledging that it marks the Taylor expansion of $S[q]$ w.r.t. $y(t)$ at $q_c(t)$, the classical trajectory, where all terms above the quadratic one vanish due to our particular form of Lagrangian, which contains only quadratic terms in $q(t)$ and $\dot{q}(t)$, such that

$$\frac{\delta^3 S[q]}{\delta q(t)\,\delta q(t')\,\delta q(t'')} = \frac{\delta}{\delta q(t)}\left(-m\frac{d^2}{d^2t'}\delta(t'-t'') - m\omega^2\delta(t'-t'')\right) = 0.$$

In any case, given eq. (4), it is easy to deduce eq. (6). Since the term linear in $y(t)$ vanishes due to $\frac{\delta S[q]}{\delta q(t)} = 0$, we get

$$S[q] = S[q_c] + \frac{1}{2}\int_0^T dt \int_0^T dt' \left(-m \frac{d^2}{dt^2}\delta(t-t') - m\omega^2 \delta(t-t')\right) y(t)\, y(t')$$

$$= S[q_c] \underbrace{- \frac{1}{2} m \int_0^T dt' \left(\frac{d}{dt}\delta(t-t')\right) y(t)\, y(t')\Big|_0^T}_{0,\ \text{since } y(0)=y(T)=0} + \frac{1}{2}\int_0^T dt \int_0^T dt' \left(+m \frac{d}{dt}\delta(t-t')\, \dot{y}(t)\, y(t') - m\omega^2 \delta(t-t')\, y(t)\, y(t')\right)$$

$$= S[q_c] + \frac{1}{2}\int_0^T dt \int_0^T dt' \left(-m\, \delta(t-t')\, \ddot{y}(t)\, y(t') - m\omega^2 \delta(t-t')\, y(t)\, y(t')\right)$$

$$= S[q_c] - \frac{m}{2}\int_0^T dt \left(\ddot{y}(t) + \omega^2 y(t)\right) y(t) = S[q_c] \underbrace{- \frac{m}{2}\dot{y}(t)\, y(t)\Big|_0^T}_{0} + \underbrace{\frac{m}{2}\int_0^T dt \left(\dot{y}^2(t) - \omega^2 y^2(t)\right)}_{= S[y]}.$$

From this result, we see that $\int Dq\, e^{iS[q]} = e^{iS[q_c]} \int Dy\, e^{iS[y]}$

c) Show that

$$S[q_c] = \frac{m\omega}{2\sin(\omega T)}\left[(q_b^2 + q_a^2)\cos(\omega T) - 2 q_a q_b\right]. \qquad (8)$$

Hint: $T$ is not connected to $\omega$ by a relation of the sort $T \propto \frac{1}{2\pi\omega}$.

The classical trajectory, $q_c(t)$, is the one that extremizes the action $S[q]$. As shown in eq. (2), for $\frac{\delta S[q]}{\delta q(t)}\Big|_{q(t)=q_c(t)} \stackrel{!}{=} 0$, it holds that

$$\ddot{q}_c(t) = -\omega^2 q_c(t),$$

of which the general solution is $q_c(t) = A \sin(\omega(t-t_0)) + B\cos(\omega(t-t_0))$. Since the harmonic oscillator is time-translation invariant, we may set $t_0 = 0$ and use the boundary conditions for $q(0)$ and $q(T)$ to determine $A$ and $B$

$$q_a := q(t=0) = A\sin(0) + B\cos(0) = B,$$

$$q_b := q(t=T) = A\sin(\omega T) + q_a\cos(\omega T) \implies A = \frac{q_b - q_a\cos(\omega T)}{\sin(\omega T)}$$

For the case of our harmonic oscillator, the classical path $q_c(t)$ is hence given by

$$q_c(t) = \frac{q_b - q_a\cos(\omega T)}{\sin(\omega T)}\sin(\omega t) + q_a\cos(\omega t)$$

Inserting the classical path into the Lagrangian and integrating from $0$ to $T$ yields

$$S[q_c] = \int_0^T dt\, \mathcal{L}\Big|_{q(t)=q_c(t)} = \int_0^T dt \left(\frac{m}{2}\dot{q}_c^2(t) - \frac{m}{2}\omega^2 q_c^2(t)\right)$$

$$= \int_0^T dt \left(\frac{m}{2}\omega^2\Big(A\cos(\omega t) - B\sin(\omega t)\Big)^2 - \frac{m}{2}\omega^2\Big(A\sin(\omega t) + B\cos(\omega t)\Big)^2\right)$$

$$= \frac{m}{2}\omega^2 \int_0^T dt \Big(A^2\cos^2(\omega t) + 2AB\sin(\omega t)\cos(\omega t) + B^2\sin^2(\omega t) - A^2\sin^2(\omega t)$$
$$- 2AB\sin(\omega t)\cos(\omega t) - B^2\cos^2(\omega t)\Big)$$

$$= \frac{m}{2}\omega^2 \int_0^T dt \Big(A^2\big(\cos^2(\omega t) - \sin^2(\omega t)\big) - B^2\big(\cos^2(\omega t) - \sin^2(\omega t)\big) - 2AB\sin(2\omega t)\Big),$$

where we used that $\sin(\alpha)\cos(\beta) \pm \sin(\beta)\cos(\alpha) = \sin(\alpha \pm \beta)$. In the next step, we'll employ $\cos(\alpha)\cos(\beta) \mp \sin(\alpha)\sin(\beta) = \cos(\alpha \pm \beta)$ to get

$$S[q_c] = \frac{m}{2}\omega^2 \int_0^T dt \Big((A^2 - B^2)\cos(2\omega t) - 2AB\sin(2\omega t)\Big)$$

$$= \frac{m}{2}\omega^2 \left(\frac{A^2 - B^2}{2\omega}\sin(2\omega t)\Big|_0^T + \frac{2AB}{2\omega}\cos(2\omega t)\Big|_0^T\right)$$

$$= \frac{m}{4}\omega\Big((A^2 - B^2)\sin(2\omega T) + 2AB\underbrace{\big(\cos(2\omega T) - 1\big)}_{2\sin^2(\omega T)}\Big),$$

where $A^2 - B^2 = \dfrac{q_b^2 - 2q_a q_b\cos(\omega T) + q_a^2\cos^2(\omega T)}{\sin^2(\omega T)} - q_a^2$, $AB = \dfrac{q_a q_b - q_a^2\cos(\omega T)}{\sin(\omega T)}$

$$S[q_c] = \frac{m\omega}{2\sin(\omega T)}\left(\frac{q_b^2 - 2q_a q_b\cos(\omega T) + q_a^2\big(\cos^2(\omega T) - \sin^2(\omega T)\big)}{2\sin(\omega T)\sin^{-1}(2\omega T)} + 2\big(q_a q_b - q_a^2\cos(\omega T)\big)\sin(\omega T)\right)$$

$$= \frac{m\omega}{2\sin(\omega T)}\left(\frac{1}{2\sin(\omega T)}\overbrace{\Big(\big(q_b^2 - 2q_a q_b\cos(\omega T) + q_a^2(\cos^2(\omega T) - \sin^2(\omega T))\big)}^{\sin(2\omega T)} + 4\big(q_a q_b - q_a^2\cos(\omega T)\big)\sin^2(\omega T)\Big)\right)$$

$$= \frac{m\omega}{2\sin(\omega T)}\Big((q_a^2 + q_b^2)\cos(\omega T) - 2q_a q_b\Big)$$

d) At this point, it is convenient to introduce functions

$$y_n(t) = C_n \sin\left(\frac{n\pi t}{T}\right), \qquad (9)$$

where $C_n$ are constants to be determined. The $y_n(t)$ are orthonormal on the interval $[0, T]$:

$$\int_0^T dt\, y_n(t)\, y_m(t) = \delta_{nm}.$$

These can be used as a basis to expand any function $y(t)$, satisfying our boundary conditions, i.e.

$$y(t) = \sum_{n=1}^{\infty} a_n y_n(t), \tag{10}$$

by means of a set of constants $a_n$. Then show that

$$S[y] = \frac{m}{2} \sum_{n=1}^{\infty} \lambda_n a_n^2 \tag{11}$$

and determine the constant quantities $\lambda_n$.

The $C_n$ are constrained by the required orthonormality.

$$\int_0^T dt\, y_n(t) y_m(t) = \int_0^T dt\, C_n \sin\left(\frac{n\pi t}{T}\right) C_m \sin\left(\frac{m\pi t}{T}\right) = \int_0^T dt\, C_n C_m \frac{1}{2}\left(\cos\left((n-m)\frac{\pi t}{T}\right) - \cos\left((n+m)\frac{\pi t}{T}\right)\right)$$

$$= \frac{1}{2} C_n C_m \left(\frac{T}{\pi(n-m)} \sin\left((n-m)\frac{\pi t}{T}\right) - \frac{T}{\pi(n+m)} \sin\left((n+m)\frac{\pi t}{T}\right)\right)\Bigg|_0^T$$

$$= \frac{T}{2\pi} C_n C_m \left(\frac{\sin((n-m)\pi)}{n-m} - \underbrace{\frac{\sin((n+m)\pi)}{n+m}}_{0}\right) = \frac{T}{2\pi} C_n C_m \frac{\sin((n-m)\pi)}{n-m} \overset{!}{=} \delta_{nm} = \begin{cases} 1 & \text{if } n=m \\ 0 & \text{else} \end{cases}$$

Since $\lim_{n\to m} \frac{\sin((n-m)\pi)}{n-m} = 1$, $C_n$ needs to fulfill $\frac{T}{2\pi} C_n^2 = 1$ or $C_n = \sqrt{\frac{2\pi}{T}}$. Therefore, our expansions of $y(t)$ and $\ddot{y}(t)$ read

$$y(t) = \sum_{n=1}^{\infty} a_n y_n(t) = \sum_{n=1}^{\infty} a_n C_n \sin\left(\frac{n\pi t}{T}\right) = \sum_{n=1}^{\infty} a_n \sqrt{\frac{2\pi}{T}} \sin\left(\frac{n\pi t}{T}\right)$$

$$\ddot{y}(t) = \sum_{n=1}^{\infty} a_n \ddot{y}_n(t) = \sum_{n=1}^{\infty} -a_n C_n \frac{n^2\pi^2}{T^2} \sin\left(\frac{n\pi t}{T}\right) = \sum_{n=1}^{\infty} -a_n \sqrt{\frac{2\pi}{T}} \frac{n^2\pi^2}{T^2} \sin\left(\frac{n\pi t}{T}\right) = -\sum_{n=1}^{\infty} a_n \frac{n^2\pi^2}{T^2} y_n(t)$$

Inserting this into the action of quantum fluctuations, $S[y]$, we get

$$S[y] = \frac{m}{2} \int_0^T dt \left(\dot{y}^2(t) - \omega^2 y^2(t)\right) = -\int_0^T dt \left(\ddot{y}(t) + \omega^2 y(t)\right) y(t)$$

$$= -\frac{m}{2} \int_0^T dt \sum_{n,m=1}^{\infty} \left(-a_n \frac{n^2\pi^2}{T^2} y_n(t) + \omega^2 a_n y_n(t)\right) a_m y_m(t)$$

$$= -\frac{m}{2} \sum_{n,m=1}^{\infty} \left(-a_n a_m \frac{n^2\pi^2}{T^2} \delta_{nm} + a_n a_m \omega^2 \delta_{nm}\right) = \frac{m}{2} \sum_{n=1}^{\infty} \left(a_n^2 \frac{n^2\pi^2}{T^2} + a_n^2 \omega^2\right)$$

$$= \frac{m}{2} \sum_{n=1}^{\infty} \underbrace{\left(\frac{n^2\pi^2}{T^2} + \omega^2\right)}_{\lambda_n} a_n^2$$

e) The integral measure can be expressed as (accept the following as a postulate, but think about it)

$$\mathcal{D}y = J \prod_{n=1}^{\infty} da_n, \tag{12}$$

for some constant $J$. Knowing this, show that

$$F_\omega(T) = \int \mathcal{D}y \, e^{iS[y]} = J \prod_{n=1}^{\infty} \sqrt{\frac{2\pi i}{m\lambda_n}}. \tag{13}$$

The above equality can be computed using the Gaussian integral.

$$F_\omega(T) = \int \mathcal{D}y \, e^{iS[y]} = J \int \prod_{n=1}^{\infty} da_n \, e^{\frac{m}{2}\sum_{n=1}^{\infty} \lambda_n a_n^2} = J \int \prod_{n=1}^{\infty} da_n \, e^{i\frac{m}{2}\lambda_n a_n^2}$$

$$= J \prod_{n=1}^{\infty} \int da_n \, e^{\underbrace{i\frac{m}{2}\lambda_n}_{-\alpha_n} a_n^2} = J \prod_{n=1}^{\infty} \sqrt{\frac{\pi}{\alpha_n}} = J \prod_{n=1}^{\infty} \sqrt{\frac{\pi}{-i\frac{m}{2}\lambda_n}} = J \prod_{n=1}^{\infty} \sqrt{\frac{2\pi i}{m\lambda_n}}$$

f) We know the exact value of $F_\omega(T)$ for the case of free fields, $\omega = 0$: Recall indeed that in this case $F_0(T) = \sqrt{\frac{m}{2\pi i T}}$. On the other hand, one can also calculate $F_0(T)$ by the same procedure we developed until now: Show that the $\lambda_n$ coefficients, when $\omega = 0$, read $\lambda_n^{(0)} = \frac{n^2\pi^2}{T^2}$

We looked at free fields ($\omega = 0$) just before part a) of this problem and found that

$$F_0(T) = \int \mathcal{D}y \, e^{iS[y]} \overset{\text{eq.(7)}}{=} e^{-iS[q_{c}]} \int \mathcal{D}q \, e^{iS[q]} = e^{-iS[q_c]} \langle q_b, T | q_a, 0 \rangle$$

$$= e^{-iS[q_c]} \langle q_b | e^{-i\hat{H}T} e^{i\hat{H}0} | q_a \rangle = e^{-iS[q_c]} \langle q_b | e^{-i\frac{\hat{p}^2}{2m}T} | q_a \rangle$$

$$= e^{-iS[q_c]} \sqrt{\frac{m}{2\pi i T}} \, e^{\underbrace{i\frac{m}{2T}(q_a - q_b)^2}_{S[q_c]|_{\omega=0},\ \text{see eq. (8)}}} = \sqrt{\frac{m}{2\pi i T}}$$

Since $\lambda_n = \frac{n^2\pi^2}{T^2} - \omega^2$, we immediately have $\lambda_n^{(0)} = \lambda_n|_{\omega=0} = \frac{n^2\pi^2}{T^2}$.

g) Then we can write

$$\frac{F_\omega(T)}{F_0(T)} = \prod_{n=1}^{\infty} \sqrt{\frac{\lambda_n^{(0)}}{\lambda_n}} = \prod_{n=1}^{\infty} \left(1 - \frac{\omega^2 T^2}{\pi^2 n^2}\right)^{-\frac{1}{2}}. \tag{14}$$

Deduce from this that

$$F_\omega(T) = \sqrt{\frac{m\omega}{2\pi i \sin(\omega T)}} \tag{15}$$

and finally, collect everything and write up the result for the transition amplitude!
We first investigate the infinite product in eq. (14). Interestingly, it vanishes, whenever $T = \frac{\pi n}{\omega}$ with $n \in \mathbb{N}$. The same applies to $\sin(\omega T)$ for $T = \frac{\pi n}{\omega}$. However, for $T \to 0$, $\prod_{n=1}^{\infty}\left(1 - \frac{\omega^2 T^2}{\pi^2 n^2}\right) \to 1$ while $\sin(\omega T) \to 0$. So the infinite product might rather resemble $\frac{\sin(\omega T)}{\omega T}$ which we stated earlier behaves as $\frac{\sin(\omega T)}{\omega T} \to 1$ for $T \to 0$. In fact, $\mathrm{sinc}(\omega T) = \frac{\sin(\omega T)}{\omega T}$ is called the cardinal sine and the infinite product is exactly Euler's expansion of it. The identity

$$\mathrm{sinc}(\omega T) = \frac{\sin(\omega T)}{\omega T} = \prod_{n=1}^{\infty}\left(1 - \frac{\omega^2 T^2}{\pi^2 n^2}\right)$$

can be proven rigorously (and lengthily) using the residue theorem (or Liouville's theorem). We refrain from performing either proof and simply apply it to find

$$F_\omega(T) = F_0(T) \prod_{n=1}^{\infty}\left(1 - \frac{\omega^2 T^2}{\pi^2 n^2}\right)^{-\frac{1}{2}} = F_0(T) \frac{1}{\sqrt{\mathrm{sinc}(\omega T)}} = \sqrt{\frac{m}{2\pi i T}} \sqrt{\frac{\omega T}{\sin(\omega T)}} = \sqrt{\frac{m\omega}{2\pi i \sin(\omega T)}}$$

Assembling everything we learned throughout this problem, we are finally able to write down the complete transition amplitude for the quantum harmonic oscillator from state $q_a$ at $t=0$ to state $q_b$ at $t=T$:

$$\langle q_b, T | q_a, 0\rangle = \int_{q_a}^{q_b} \mathcal{D}q \, e^{iS[q]} = e^{iS[q_c]} \int \mathcal{D}y \, e^{iS[y]} = e^{iS[q_c]} F_\omega(T)$$

$$= \sqrt{\frac{m\omega}{2\pi i \sin(\omega T)}} \, e^{\frac{im\omega}{2\sin(\omega T)}\left((q_b^2 + q_a^2)\cos(\omega T) - 2q_a q_b\right)}$$