Quantum Field Theory II
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# Assignment 6

Due: Week beginning 25.05.2015.

**Problem 6.1 (Path integral of the fermionic field with sources):**
We define the generating functional for the correlation functions of the free Dirac theory as

$$Z_0[\bar{\eta},\eta] = \int \mathcal{D}\bar{\psi}\mathcal{D}\psi\, e^{i(S_0[\bar{\psi},\psi]+\bar{\eta}\cdot\psi+\bar{\psi}\cdot\eta)} \tag{1}$$

with

$$S_0[\bar{\psi},\psi] = \int d^4x\, \bar{\psi}(i\partial\!\!\!/ - m_0)\psi\,.$$

a) Show by means of 'completing the square' that (1) can be rearranged to

$$Z_0[\bar{\eta},\eta] = Z_0[\bar{\eta}=0,\eta=0]\, e^{-\bar{\eta}\cdot S_F\cdot\eta}\,, \tag{2}$$

where $S_F$ is the Feynman propagator of the free Dirac theory, i.e.

$$(i\partial\!\!\!/_x - m_0 + i\epsilon)\, S_F(x-y) = i\delta^{(4)}(x-y)$$

**Hint**: You may use that the measure is invariant under the (fermionic) coordinate change you have to apply to find (2).

b) We add now an interaction part $\int d^4x\, \mathcal{L}_{\text{int}}(\bar{\psi},\psi)$ to $S_0[\bar{\psi},\psi]$. By proceeding as in the lecture for the bosonic case, show that the generating functional for the interacting theory can be written as:

$$Z[\bar{\eta},\eta] = Z_0[\bar{\eta}=0,\eta=0]\, e^{-\frac{\delta}{\delta\psi}\cdot S_F\cdot\frac{\delta}{\delta\bar{\psi}}}\, e^{i\int d^4x\, \mathcal{L}_{\text{int}}(\bar{\psi},\psi)+i\bar{\eta}\cdot\psi+i\bar{\psi}\cdot\eta}\Big|_{\bar{\psi}=\psi=0}\,. \tag{3}$$

**Hint**: You might find the following identity useful:

$$F\left[\frac{\delta}{i\delta\bar{\eta}}\right] G\left[\bar{\eta}\right]\, e^{i\,\bar{\eta}\cdot\psi} = (-1)^{\deg(F)\deg(G)}\, G\left[-\frac{\delta}{i\delta\psi}\right] F\left[\psi\right]\, e^{i\,\bar{\eta}\cdot\psi}\,, \tag{4}$$

where $F$ and $G$ are either Grassmann odd or even, i.e. their degree (grade) is either zero or one. If you use (4), you should also prove it.

**Problem 6.2 (Wick's theorem for fermions):**
From (3) it is obvious that the correlation functions for the free theory are given by

$$\langle T \prod_{i=1}^{n} \psi(x_i) \prod_{j=n+1}^{m+n} \bar{\psi}(x_j) \rangle = e^{-\frac{\delta}{\delta \psi} \cdot S_F \cdot \frac{\delta}{\delta \bar{\psi}}} \prod_{i=1}^{n} \psi(x_i) \prod_{j=n+1}^{m+n} \bar{\psi}(x_j) \Big|_{\bar{\psi}=\psi=0} . \tag{5}$$

a) Prove Wick's theorem for the free fermions, i.e. show that $\langle T \prod_{i=1}^{n} \psi(x_i) \prod_{j=n+1}^{m+n} \bar{\psi}(x_j) \rangle = 0$ for $n \neq m$ and

$$\begin{aligned} \langle T \prod_{i=1}^{n} \psi(x_i) \prod_{j=n+1}^{m+n} \bar{\psi}(x_j) \rangle =& (-1)^{(n-1)\frac{n}{2}} S_F(x_1 - x_{n+1}) S_F(x_2 - x_{n+2}) \dots S_F(x_n - x_{2n}) + \\ & + \text{all other contractions with appropriate signs.} \end{aligned} \tag{6}$$

for $n = m$.

b) Show by means of (6) that

$$\begin{aligned} \langle T \bar{\psi}_A(x) M^A{}_B \psi^B(x) \bar{\psi}_C(y) \tilde{M}^C{}_D \psi^D(y) \rangle =& - \operatorname{Tr} \left( S_F(y-x) M S_F(x-y) \tilde{M} \right) + \\ & + \text{`uninteresting' } S_F(0)\text{-terms} . \end{aligned}$$

**Note**: From this one obtains the Feynman rule that one has to add a minus sign for every fermionic loop.