Assignment 10

Due: Week beginning 29.06.2015.

Problem 10.1 (One loop β-function in QED):

a) Compute the one-loop β-function in QED.
Hint: You should find $\beta_e = \frac{e^3}{12\pi^2}$. Start from the the general expression

$$\beta_e(\mu) = \mu \frac{\partial}{\partial \mu} \left(-e\,\delta_1 + e\,\delta_2 + \tfrac{1}{2}\,e\,\delta_3 \right) \tag{1}$$

together with the one-loop counterterms derived in QFT I:

$$\begin{aligned}
\delta_1 &= -\frac{e^2}{(4\pi)^{d/2}} \int_0^1 dz\,(1-z) \Big(\frac{\Gamma(2-\frac{d}{2})}{((1-z)^2 m^2 + z\,M^2)^{2-d/2}} \frac{(2-\epsilon)^2}{2} + \\
&\quad + \frac{\Gamma(3-\frac{d}{2})}{((1-z)^2 m^2 + z\,M^2)^{3-d/2}} (2(1-4z+z^2) - \epsilon(1-z)^2) m^2 \Big)\,, \\
\delta_2 &= -\frac{e^2}{(4\pi)^{d/2}} \int_0^1 dz\, \frac{\Gamma(2-\frac{d}{2})}{((1-z)^2 m^2 + z\,M^2)^{2-d/2}} \big((2-\epsilon) z + \\
&\quad - \frac{\epsilon}{2} \frac{2\,z\,(1-z)\,m^2}{(1-z)^2 m^2 + z\,M^2} (4 - 2\,z - \epsilon\,(1-z)) \big)\,, \\
\delta_3 &= -\frac{e^2}{(4\pi)^{d/2}} \int_0^1 dz \frac{\Gamma(2-\frac{d}{2})}{(m^2)^{2-d/2}} (8\,z\,(1-z))\,,
\end{aligned} \tag{2}$$

where M was just an IR cutoff and m the electron mass. Consider what plays the role of the renormalisation scale μ in the renormalisation scheme underlying these results from QFT I.

b) Integrate the solution to the renormalisation group equation $\frac{d}{d\log\mu} e(\mu) = \beta_e(\mu)$ to find at one-loop order

$$\alpha(\mu) = \frac{\alpha^*}{1 - \frac{2}{3\pi}\,\alpha^* \log \frac{\mu}{\mu^*}}, \qquad \alpha := \frac{e^2}{4\pi} \tag{3}$$

where $\alpha^* = \alpha(\mu^*)$ with μ^* some fixed scale. Plot the behaviour of $e(\mu)$. At which scale does e cease to be perturbative?

Problem 10.2 (Anomalous dimension):

a) Consider the two-point function $G_2(p;\lambda,\mu)$ of a massless scalar theory and rewrite it as

$$G_2(p;\lambda,\mu) = \frac{i}{p^2} f(-\frac{p^2}{\mu^2}) \tag{4}$$

with f some function. Show that the Callan-Symanzik equation

$$\Big(\mu \frac{\partial}{\partial \mu} + \beta(\lambda) \frac{\partial}{\partial \lambda} + 2\gamma(\lambda) \Big) G_2(p;\lambda,\mu) = 0 \tag{5}$$

can be cast into the form

$$\Big(p\frac{\partial}{\partial p} - \beta(\lambda)\frac{\partial}{\partial \lambda} + 2 - 2\gamma(\lambda)\Big)G_2(p;\lambda,\mu) = 0, \tag{6}$$

where $p = \sqrt{-p^2}$.

b) Integrate this equation to find

$$G_2(p;\lambda,\mu) = \frac{i}{p^2}\mathcal{F}(\bar{\lambda}(p;\lambda))\exp\left(2\int_{p'=\mu}^{p} d\left[\log\tfrac{p'}{\mu}\right]\gamma\left(\bar{\lambda}(p';\lambda)\right)\right), \tag{7}$$

where $\bar{\lambda}(p;\lambda)$ is the *running coupling*, which is defined such that $p\frac{\partial}{\partial p}\bar{\lambda}(p;\lambda) = \beta(\bar{\lambda}(p;\lambda))$ with the initial condition $\bar{\lambda}(\mu;\lambda) = \lambda$, and $\mathcal{F}(x)$ is some a priori unknown function.
Hint: Compare (6) with

$$\left(p\frac{d}{dp} + 2 - 2\gamma(\lambda)\right) G(p;\bar{\lambda}(\tfrac{\mu^2}{p};\lambda^*),\mu)\Big|_{\lambda^*=\bar{\lambda}(p;\lambda)} = 0\,, \tag{8}$$

and use your knowledge about first order differential equations to integrate (8) and, therefore, to obtain (7). Note that $\bar{\lambda}(\frac{\mu^2}{p};x)$ is the inverse of $\bar{\lambda}(p;x)$, i.e. $\bar{\lambda}\left(\frac{\mu^2}{p};\bar{\lambda}(p;x)\right) = x$.

c) Argue that in the vicinity of a fixed point λ^* where $\beta(\lambda^*) = 0$ the two-point function scales as

$$G_2(p;\lambda^*,\mu) \propto \left(\frac{1}{p^2}\right)^{1-\gamma(\lambda^*)} \tag{9}$$

and justify from this the term *anomalous dimension* for γ.