String Theory

Solution to Assignment 8

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1 The free boson on the sphere, normal ordering, and all that

Consider the action for the free boson

$$S[X] = \frac{1}{2\pi\alpha'} \int \mathrm{d}z\mathrm{d}\bar{z}\, \partial_z X(z,\bar{z})\, \partial_{\bar{z}} X(z,\bar{z}), \tag{1}$$

with energy momentum tensor

$$T(z) = -\frac{1}{\alpha'} \mathcal{N}[\partial X_c(z) \partial X_c(z)], \tag{2}$$

in terms of the (anti-)chiral fields $X(z,\bar{z}) = X_c(z) + X_a(\bar{z})$.

a) Recall from the lecture the definition of the normal ordering prescription $\mathcal{N}$ employed above. Give the general relation between normal ordered and radially ordered operators.

b) Give the correlator $\langle X_c(z) X_c(w) \rangle$ for the field $X_c(z)$. Compare this with the correlator of two primary fields in a general CFT. Deduce the correlator $\langle \partial_z X_c(z) \partial_w X_c(w) \rangle$ from $\langle X_c(z) X_c(w) \rangle$.

c) Review the derivation of the OPE of $T(z)\, \partial X_c(w)$ for the primary field $\partial X_c(w)$ via Wick's theorem as given in the lecture. Following that same logic, compute the OPE

$$\mathcal{R}[T(z)\, T(w)] = \frac{1}{\alpha'^2} \mathcal{N}[\partial X_c(z) \partial X_c(z)] \mathcal{N}[\partial X_c(w) \partial X_c(w)], \tag{3}$$

by writing down all cross-contractions between fields not in the same normal ordered expression and bring the result into the standard form

$$T(z) T(w) = \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + \text{(finite terms)}. \tag{4}$$

Take a minute to enjoy the ease of this derivation of the normal ordering constant c as compared to a brute-force computation in terms of the modes.

d) Now consider the normal ordered field $\mathcal{N}(e^{ikX_c(z)})$. By expanding the exponential $\mathcal{N}(e^{ikX_c(z)}) = \sum_{n=0}^{\infty} \frac{(ik)^n}{n!} \mathcal{N}[X_c^n(z)]$ and using Wick's theorem, derive the OPE

$$\mathcal{R}[\partial X_c(z)\, \mathcal{N}(e^{ikX_c(w)})] = -\frac{i\alpha' k}{2} \frac{\mathcal{N}(e^{ikX_c(w)})}{z-w} + \text{(finite terms)}. \tag{5}$$

Use this to compute also the OPE

$$\mathcal{R}[T(z)\, \mathcal{N}(e^{ikX_c(w)})] = \frac{\alpha' k^2}{4} \frac{\mathcal{N}(e^{ikX_c(w)})}{(z-w)^2} + \frac{\partial \mathcal{N}(e^{ikX_c(w)})}{z-w}. \tag{6}$$

What does this tell us about the nature of $\mathcal{N}(e^{ikX_c(z)})$?

a) **Normal ordering** is the prescription of moving all creation operators to the left, so that the annihilation operators act first. Actually choosing which modes represent creators and which annihilators usually goes hand in hand with the choice of vacuum a particular $|\Omega\rangle$, where the goal always is to match the two in such a way that the vacuum expectation value of any normal-ordered product of operators is zero,

$$\langle\Omega|\mathcal{N}[\textstyle\prod_{i=1}^n \hat{\mathcal{O}}_i]|\Omega\rangle = 0. \tag{7}$$

In string theory, we identify modes α_m^μ with $m > 0$ as annihilation and those with $m \leq 0$ as creation operators. Normal ordering can then be implemented via the map $\mathcal{N}$,

$$\mathcal{N}(\alpha_m^\mu\,\alpha_n^\nu) = \begin{cases} \alpha_m^\mu\,\alpha_n^\nu & \text{for } m \leq n, \\ \alpha_n^\nu\,\alpha_m^\mu & \text{for } n < m. \end{cases} \tag{8}$$

Another thing that should not go unmentioned in any discussion of normal ordering is **Wick's theorem**. It is used extensively in (perturbative) quantum field theories to reduce arbitrary products of creators and annihilators to sums of products of pairs of these operators. In string theory, it relates normal- with time-/radially-ordered products of fields, where time-ordering applies to the cylinder and radial-ordering arises on the complex plane respectively the Riemann sphere. For a product of just two radially-ordered fields $\phi_1(z)$, $\phi_2(w)$, Wick's theorem states

$$\mathcal{R}[\phi_1(z)\,\phi_2(w)] = \mathcal{N}[\phi_1(z)\,\phi_2(w)] + \langle\Omega|\phi_1(z)\,\phi_2(w)|\Omega\rangle. \tag{9}$$

This relation can be inductively used to relate time-ordered and normal-ordered products of more than two fields by summing over all possible contractions, i.e. by successively taking all pairs of fields and replacing them by their two-point correlator. E.g. for three fields $\phi_1(z)$, $\phi_2(w)$, and $\phi_3(z_3)$, Wick's theorem would read

$$\begin{aligned} \mathcal{R}[\phi_1(z)\,\phi_2(w)\,\phi_3(z_3)] = \mathcal{N}\Big[&\phi_1(z)\,\phi_2(w)\,\phi_3(z_3) + \phi_1(z)\langle\Omega|\phi_2(w)\,\phi_3(z_3)|\Omega\rangle \\ &+ \phi_2(z)\langle\Omega|\phi_1(w)\,\phi_3(z_3)|\Omega\rangle + \phi_3(z)\langle\Omega|\phi_1(w)\,\phi_2(z_3)|\Omega\rangle\Big] \\ = \mathcal{N}[&\phi_1(z)\,\phi_2(w)\,\phi_3(z_3)] + \sum_{i\neq j\neq k=1}^{3} \phi_i(z_i)\langle\Omega|\phi_j(z_j)\,\phi_k(z_k)|\Omega\rangle. \end{aligned} \tag{10}$$

Note: The normal-ordered product of two operators is precisely the non-singular part in the an OPE. That is to say if $\phi_1(z)$, $\phi_2(w)$ denote two fields, then their radially-ordered product expansion decomposes into the two terms

$$\mathcal{R}[\phi_1(z)\,\phi_2(w)] = \underbrace{\sum_{n=1}^{\infty} \frac{a_n}{(z-w)^n}}_{\text{singular piece}} + \mathcal{N}[\phi_1(z)\,\phi_2(w)], \tag{11}$$

which in turn, comparing with eq. (9), identifies the singular piece with the vacuum expectation value.

b) Rather than just give the correlator $\langle X_c(z)X_c(w)\rangle$, we will try to derive it here starting from the free boson action (1). First, note that since

$$\begin{aligned} 0 \overset{!}{=} \frac{\delta S[X]}{\delta X(z,\bar{z})} &= \frac{1}{\pi\alpha'}\int \mathrm{d}z'\mathrm{d}\bar{z}'\,\partial_{z'}X(z',\bar{z}')\,\frac{\partial_{\bar{z}'}\delta X(z',\bar{z}')}{\delta X(z,\bar{z})} \\ &= -\frac{1}{\pi\alpha'}\int \mathrm{d}z'\mathrm{d}\bar{z}'\,\partial_{\bar{z}'}\partial_{z'}X(z',\bar{z}')\,\delta(z-z')\,\delta(\bar{z}-\bar{z}') = -\frac{1}{\pi\alpha'}\partial_{\bar{z}}\partial_z X(z,\bar{z}), \end{aligned} \tag{12}$$

the string field's e.o.m. on the plane is given by $\partial_z\partial_{\bar{z}}X(z,\bar{z}) = 0$ which $X(z,\bar{z}) = X_c(z) + X_a(\bar{z})$ is indeed the general solution of, and entirely equivalent to the e.o.m.s $(\partial_\tau^2 - \partial_\sigma^2)X(\tau,\sigma) = 0$

and $\partial_+\partial_- X(\xi^+,\xi^-) = 0$ we found for regular worldsheet coordinates and in lightcone gauge, respectively.[1] Using the e.o.m., we can obtain the two-point function directly from the path integral,

$$
\begin{aligned}
0 &= \frac{1}{Z}\int \mathcal{D}X(z',\bar z')\, \frac{\delta}{\delta X(z,\bar z)}\Big(X(z',\bar z')e^{-S[X(z',\bar z')]}\Big) \\
&= \frac{1}{Z}\int \mathcal{D}X(z',\bar z')\Big(\frac{\delta X(z',\bar z')}{\delta X(z,\bar z)}e^{-S[X(z',\bar z')]} - X(z',\bar z')e^{-S[X(z',\bar z')]}\frac{\delta S[X(z',\bar z')]}{\delta X(z,\bar z)}\Big) \\
&\overset{(12)}{=} \frac{1}{Z}\int \mathcal{D}X(z',\bar z')\, e^{-S[X(z',\bar z')]}\Big(\delta(z-z')\,\delta(\bar z-\bar z') + \frac{1}{\pi\alpha'}X(z',\bar z')\partial_{\bar z}\partial_z X(z,\bar z)\Big) \\
&= \Big\langle \delta(z-z')\,\delta(\bar z-\bar z') + \frac{1}{\pi\alpha'}X(z',\bar z')\partial_{\bar z}\partial_z X(z,\bar z)\Big\rangle,
\end{aligned}
\tag{13}
$$

where the very first equality $0 = \dots$ comes from the fact that the path integral over a manifold M of a functional derivative $\frac{\delta F[\phi(x)]}{\delta\phi(x)}$ gives just the function $F[\phi(x)]$ evaluated at the boundary ∂M, i.e.

$$
\int_M \mathcal{D}\phi(x)\,\frac{\delta F[\phi(x)]}{\delta\phi(x)} = F[\phi(x)]\Big|_{\partial M}, \tag{14}
$$

which we take to be zero in the case of the localized string field. Thus, we found

$$
\partial_{\bar z}\partial_z\langle X(z,\bar z)X(w,\bar w)\rangle = -\pi\alpha'\delta(z-w)\,\delta(\bar z-\bar w). \tag{15}
$$

To make this more explicit, we revert to Stokes theorem on the complex plane,

$$
\int_U \Big(\partial_{\bar z}F - \partial_z G\Big)\mathrm{d}z\mathrm{d}\bar z = -i\oint_{\partial U}\Big(F\,\mathrm{d}z + G\,\mathrm{d}\bar z\Big), \tag{16}
$$

where $F = F(z,\bar z)$, $G = G(z,\bar z)$ are continuously differentiable functions on an open region U of $\mathbb{C}$. For the special case of $G = 0$, $F = \frac{1}{z}$, and $U = B_r(0)$ a ball of radius r at the origin, we get

$$
\int_{B_r(0)} \partial_{\bar z}\frac{1}{z}\mathrm{d}z\mathrm{d}\bar z = -i\underbrace{\oint_{\partial B_r(0)}\frac{1}{z}\,\mathrm{d}z}_{2\pi i} = 2\pi = 2\pi\underbrace{\int\delta(z)\,\delta(\bar z)\,\mathrm{d}z\mathrm{d}\bar z}_{1}, \tag{17}
$$

from which we infer

$$
2\pi\delta(z)\,\delta(\bar z) = \partial_{\bar z}\frac{1}{z}. \tag{18}
$$

By reinserting this handy little formula and using $\partial_{\bar z}\frac{1}{z} = \partial_{\bar z}\frac{\bar z}{z\bar z} = \partial_{\bar z}\partial_z\ln(|z|^2)$, eq. (15) becomes

$$
\partial_{\bar z}\partial_z\langle X(z,\bar z)X(w,\bar w)\rangle = -\frac{\alpha_2}{2}\partial_{\bar z}\partial_z\ln(|z-w|^2), \tag{19}
$$

which we can integrate twice to obtain

$$
\langle X(z,\bar z)X(w,\bar w)\rangle = -\frac{\alpha_2}{2}\ln(|z-w|^2). \tag{20}
$$

This is the familiar result that the Green's function in two dimensions is logarithmic.

> **Note:** This logarithmic correlation exempts the string field $X(z,\bar z)$ from having a definite scaling dimension, meaning it is not a primary nor even a quasi-primary field of the free bosonic CFT. However, as we will see, both its derivative and exponential are primary fields.

[1] We did not pick up any boundary terms from the partial integration in eq. (12) because the timelike ones vanish due to the variation being held fixed at initial and final times, i.e. $\delta X(z,\bar z)|_{\tau_i,\tau_f} = 0$, and the spacelike boundary terms either cancel each other for the closed string with periodic boundaries or vanish separately for the open string with Dirichlet or Neumann boundary conditions.

Since $\langle X(z,\bar{z})X(w,\bar{w})\rangle = \langle X_c(z)X_c(w)\rangle + \langle X_c(z)X_a(\bar{w})\rangle + \langle X_a(\bar{z})X_c(w)\rangle + \langle X_a(\bar{z})X_a(\bar{w})\rangle$ and $\ln(|z-w|^2)$ can be split up into $\ln[(z-w)(\bar{z}-\bar{w})] = \ln(z-w) + \ln(\bar{z}-\bar{w})$, we simply match terms based on their functional arguments to identify the chiral and antichiral two-point correlators as

$$\langle X_c(z)X_c(w)\rangle = -\frac{\alpha'}{2}\ln(z-w), \qquad \langle X_a(\bar{z})_1X_a(\bar{w})\rangle = -\frac{\alpha'}{2}\ln(\bar{z}-\bar{w}). \tag{21}$$

From this in turn, $\langle \partial_z X_c(z)\partial_w X_c(w)\rangle$ follows by differentiation,

$$\begin{aligned}\langle \partial_z X_c(z)\partial_w X_c(w)\rangle &= \partial_z\partial_w\langle X_c(z)X_c(w)\rangle \overset{(21)}{=} -\frac{\alpha'}{2}\partial_w\partial_z\ln(z-w)\\ &= \frac{\alpha'}{2}\partial_z\frac{1}{z-w} = -\frac{\alpha'}{2}\frac{1}{(z-w)^2}\end{aligned} \tag{22}$$

> **Note:** This indeed fits the form of the two-point function of two quasi-primaries, $\langle\phi_i(z_i)\phi_j(z_j)\rangle = \frac{d_{ij}\,\delta_{h_i,h_j}}{(z_i-z_j)^{2h_i}}$, which is a sufficient condition for $\partial_z X_c(z)$ to be a quasi-primary field of weight $h=1$.

c) Applying Wick's theorem to $\mathcal{R}[T(z)\,\partial_w X_c(w)]$ gives

$$\begin{aligned}\mathcal{R}[T(z)\,\partial_w X_c(w)] &\overset{(2)}{=} -\frac{1}{\alpha'}\mathcal{R}\Big[\mathcal{N}[\partial_z X_c(z)\partial_z X_c(z)]\,\partial_w X_c(w)\Big] \\ &\overset{(10)}{=} -\frac{1}{\alpha'}\Big[\underbrace{\mathcal{N}\big[\partial_z X_c(z)\partial_z X_c(z)\,\partial_w X_c(w)\big]}_{\text{non-singular piece}} + 2\,\partial_z X_c(z)\underbrace{\langle\partial_z X_c(z)\partial_w X_c(w)\rangle}_{-\frac{\alpha'}{2}\frac{1}{(z-w)^2}} \\ &\qquad + \partial_w X_c(w)\underbrace{\langle\mathcal{N}[\partial_z X_c(z)\partial_z X_c(z)]\rangle}_{0,\text{ by eq. }(7)}\Big].\end{aligned} \tag{23}$$

Expanding the prefactor $\partial_z X_c(z)$ of the singular term around $z=w$ to first order,

$$\partial_z X_c(z)\big|_w = \partial_w X_c(w) + \partial_w^2 X_c(w)(z-w) + \mathcal{O}[(z-w)^2], \tag{24}$$

and reinserting it into (23) gives

$$\boxed{\mathcal{R}[T(z)\,\partial_w X_c(w)] = \frac{\partial_w X_c(w)}{(z-w)^2} + \frac{\partial_w^2 X_c(w)}{z-w} + (\text{terms non-singular at } z=w).} \tag{25}$$

This matches the OPE of a primary field $\phi(z)$ of weight h with the energy-momentum tensor,

$$\mathcal{R}[T(z)\,\phi(w)] = \frac{h\,\phi(w)}{(z-w)^2} + \frac{\partial_w\phi(w)}{z-w} + (\text{terms non-singular at } z=w), \tag{26}$$

identifying $\partial_w X_c(w)$ as a primary with $h=1$.

To compute the OPE of $\mathcal{R}[T(z)\,T(w)]$, we again employ Wick's theorem,

$$\begin{aligned}\mathcal{R}[T(z)\,T(w)] &\overset{(2)}{=} \frac{1}{\alpha'^2}\mathcal{R}\Big[\mathcal{N}[\partial_z X_c(z)\partial_z X_c(z)]\,\mathcal{N}[\partial_w X_c(w)\partial_w X_c(w)]\Big] \\ &\overset{(10)}{=} \frac{1}{\alpha'^2}\mathcal{N}\Big[\partial_z X_c(z)\partial_z X_c(z)\,\partial_w X_c(w)\partial_w X_c(w) \\ &\qquad + 4\,\partial_z X_c(z)\partial_w X_c(w)\langle\partial_z X_c(z)\partial_w X_c(w)\rangle \\ &\qquad + 2\,\langle\partial_z X_c(z)\partial_w X_c(w)\rangle^2\Big],\end{aligned} \tag{27}$$

where we ignored all contractions inside normal-ordered operator products, i.e.

$$\begin{aligned} & 2\,\mathcal{N}[\partial_z X_c(z)\partial_z X_c(z)]\,\langle\mathcal{N}[\partial_z X_c(z)\partial_z X_c(z)]\rangle, \\ \text{and} \quad & \langle\mathcal{N}[\partial_z X_c(z)\partial_z X_c(z)]\rangle\,\langle\mathcal{N}[\partial_w X_c(w)\partial_w X_c(w)]\rangle, \end{aligned} \tag{28}$$

by argument of eq. (7). Inserting eqs. (22) and (24) into eq. (27) yields

$$\begin{aligned} \mathcal{R}[T(z)\,T(w)] = \frac{1}{\alpha'^2}\Big[& \mathcal{N}\big[\partial_z X_c(z)\partial_z X_c(z)\,\partial_w X_c(w)\partial_w X_c(w)\big] \\ & - 2\,\alpha'\,\frac{\mathcal{N}\big[\partial_z X_c(z)\partial_w X_c(w)\big]}{(z-w)^2} + \frac{\alpha'^2}{2}\frac{1}{(z-w)^4}\Big]. \end{aligned} \tag{29}$$

Inserting the following Taylor expansion for the numerator of the second term,

$$\begin{aligned} & \mathcal{N}\big[\partial_z X_c(z)\partial_w X_c(w)\big] \\ & = \underbrace{\mathcal{N}\big[\partial_w X_c(w)\partial_w X_c(w)\big]}_{-\alpha' T(w)} + \mathcal{N}\big[[\partial_w^2 X_c(w)]\partial_w X_c(w)\big](z-w) + \mathcal{O}[(z-w)^2] \\ & = -\alpha' T(w) + \frac{1}{2}\underbrace{\partial_w \mathcal{N}\big[\partial_w X_c(w)\partial_w X_c(w)\big]}_{-\alpha'\partial_w T(w)}(z-w) + \mathcal{O}[(z-w)^2]. \end{aligned} \tag{30}$$

we finally get the OPE of the energy-momentum tensor with itself,

$$\boxed{\mathcal{R}[T(z)\,T(w)] = \frac{\overset{\frac{1}{2}\leftarrow c}{}}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial_w T(w)}{z-w} + (\text{terms non-singular at } z=w),} \tag{31}$$

which confirms that the central extension c of the Virasoro algebra is given by 1. $c \neq 0$ signals a quantum anomaly of the CFT's classical conformal symmetry.

d) Next, we compute the OPE of $\mathcal{R}[\partial_z X_c(z)\,\mathcal{N}(e^{ikX_c(w)})]$, where $\mathcal{N}(e^{ikX_c(w)})]$ can be expanded into $\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\mathcal{N}[X_c^n(w)]$.

$$\begin{aligned} & \mathcal{R}[\partial_z X_c(z)\,\mathcal{N}(e^{ikX_c(w)})] \\ & = \sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\quad \underbrace{\mathcal{R}\big[\partial_z X_c(z)\,\mathcal{N}[X_c^n(w)]\big]}_{\mathcal{N}\left[\partial_z X_c(z)\,X_c^n(w)+n\,X_c^{n-1}(w)\langle\partial_z X_c(z)X_c(w)\rangle\right],\ \text{by Wick's thm.}} \\ & \overset{\text{W.T.}}{=} \sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\mathcal{N}\big[\partial_z X_c(z)\,X_c^n(w)\big] + \sum_{n=1}^{\infty}\frac{(ik)^n}{(n-1)!}\mathcal{N}[X_c^{n-1}(w)]\,\partial_z\langle X_c(z)X_c(w)\rangle. \end{aligned} \tag{32}$$

The first term is simply the operator product we started with but now in normal-ordered form, $\mathcal{N}[\partial_z X_c(z)\,\mathcal{N}(e^{ikX_c(w)})]$. For the correlator in the second term, we adjust the sum to start at $n=0$ and insert our result $\langle X_c(z)X_c(w)\rangle = -\frac{\alpha'}{2}\ln(z-w)$ from eq. (21) to obtain

$$\begin{aligned} \mathcal{R}[\partial_z X_c(z)\,\mathcal{N}(e^{ikX_c(w)})] & = \underbrace{\mathcal{N}[\partial_z X_c(z)\,\mathcal{N}(e^{ikX_c(w)})]}_{\text{non-singular at } z=w} - ik\frac{\alpha'}{2}\sum_{n=1}^{\infty}\frac{(ik)^n n}{n!}\,\frac{\mathcal{N}[X_c^{n-1}(w)]}{z-w} \\ & = -ik\frac{\alpha'}{2}\,\frac{\mathcal{N}[e^{ikX_c(w)}]}{z-w} + (\text{terms non-singular at } z=w). \end{aligned} \tag{33}$$

We can use eq. (33) to further calculate $\mathcal{R}[T(z)\,\mathcal{N}(e^{ikX_c(w)})]$.

$$\begin{aligned}\mathcal{R}[T(z)\,\mathcal{N}(e^{ikX_c(w)})] &= -\frac{1}{\alpha'}\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\mathcal{R}\Big[\mathcal{N}[\partial_z X_c(z)\,\partial_z X_c(z)]\,\mathcal{N}[X_c^n(w)]\Big]\\ &\overset{\text{W.T.}}{=} -\frac{1}{\alpha'}\Bigg[\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\mathcal{N}[\partial_z X_c(z)\,\partial_z X_c(z)\,X_c^n(w)] \quad \leftarrow \text{non-singular}\\ &\quad + \langle\partial_z X_c(z) X_c(w)\rangle\sum_{n=1}^{\infty}\frac{(ik)^n}{n!}\,2n\,\mathcal{N}[\partial_z X_c(z)\,X_c^{n-1}(w)]\\ &\quad + \langle\partial_z X_c(z) X_c(w)\rangle^2\sum_{n=2}^{\infty}\frac{(ik)^n}{n!}\,n(n-1)\,\mathcal{N}[X_c^{n-2}(w)]\Bigg]\end{aligned} \tag{34}$$

At this point, we again use $\langle\partial_z X_c(z) X_c(w)\rangle = -\frac{\alpha'}{2}\frac{1}{z-w}$ and adjust the sums to start at $n=0$ to obtain

$$\begin{aligned}&\mathcal{R}[T(z)\,\mathcal{N}(e^{ikX_c(w)})]\\ &= -\frac{1}{\alpha'}\frac{\alpha'^2}{4}\frac{(ik)^2}{(z-w)^2}\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\mathcal{N}[X_c^n(w)] + \frac{1}{\alpha'}\frac{\alpha'}{2}\frac{2ik}{z-w}\sum_{n=0}^{\infty}\frac{(ik)^n}{n!}\mathcal{N}[\partial_z X_c(z)\,X_c^{n-1}(w)]\\ &\quad + (\text{terms non-singular at } z=w)\\ &= \frac{\alpha' k^2}{4}\frac{\mathcal{N}(e^{ikX_c(w)})}{(z-w)^2} + \frac{ik\mathcal{N}[\overbrace{\partial_z X_c(z)}^{\partial_w X_c(w)+\mathcal{O}[z-w]}\,e^{ikX_c(w)}]}{z-w} + (\text{terms non-singular at } z=w)\\ &= \frac{\alpha' k^2}{4}\frac{\mathcal{N}(e^{ikX_c(w)})}{(z-w)^2} + \frac{\partial_z\mathcal{N}[e^{ikX_c(w)}]}{z-w} + (\text{terms non-singular at } z=w).\end{aligned} \tag{35}$$

This matches the operator product expansion of a primary field with $T(z)$ as given in eq. (26) and tells us that $\mathcal{N}(e^{ikX_c(w)})$ is a primary with weight $h = \frac{\alpha' k^2}{4}$.

2 Unitary CFTs

The OPE of the energy-momentum tensor $T(z)$ with a primary field $\phi(w)$ of dimension h is

$$T(z)\,\phi(w) = \frac{h\,\phi(w)}{(z-w)^2} + \frac{\partial_w\phi(w)}{z-w} + (\text{non-singular terms}). \tag{36}$$

a) Let $\phi(w)$ be a primary field. Use eq. (36) to show the following action of the Virasoro generators L_m, defined by $T(z) = \sum_{m\in\mathbb{Z}} z^{-m-2}L_m$,

$$L_{-1}\phi(w) = \partial_w\phi(w), \qquad L_0\phi(w=0) = h\phi(w=0). \tag{37}$$

b) Use the Virasoro algebra to compute the norm $\langle\phi|L_m L_{-m}|\phi\rangle$ for the state $|\phi\rangle = \phi(0)|0\rangle$ and $m \in \mathbb{N}_0$. Deduce that if the CFT is to be unitary, the central term c of the Virasoro algebra must satisfy $c \geq 0$ and $h \geq 0$ for all primary fields.

c) Use eq. (37) to show that in a unitary CFT, a primary field of dimension $h = 0$ must be a constant field.

Note: Locality and the equal time commutation relations for a constant field imply that it must be a simple number in $\mathbb{C}$. Therefore in a unitary CFT, the only primary with $h = 0$ is the identity operator with associated highest weight state the $PSL(2,\mathbb{C})$-invariant vacuum $|0\rangle$.

a) $T(z)$ is a chiral tensor of weight $(h, \bar{h}) = (2, 0)$. Its Laurent series on the complex plane thus reads

$$T(z) = \sum_{m\in\mathbb{Z}} z^{-m-2}\, L_m, \tag{38}$$

in terms of the Virasoro generators

$$L_m = \oint_{C_0} \frac{\mathrm{d}z}{2\pi i} z^{m+1}\, T(z). \tag{39}$$

Inserting this integral representation of L_m into eq. (37)'s first equality and using eq. (36) gives

$$L_{-1}\phi(w) = \oint_{C_w} \frac{\mathrm{d}z}{2\pi i}\, T(z)\,\phi(w) \overset{(36)}{=} \oint_{C_w} \frac{\mathrm{d}z}{2\pi i} \left[\frac{h\,\phi(w)}{(z-w)^2} + \frac{\partial_w \phi(w)}{z-w}\right] = \partial_w \phi(w). \tag{40}$$

Equation (37)'s second equality may be demonstrated in the same way,

$$L_0\phi(0) = \oint_{C_0} \frac{\mathrm{d}z}{2\pi i}\, z\, T(z)\,\phi(0) \overset{(36)}{=} \oint_{C_0} \frac{\mathrm{d}z}{2\pi i} \left[\frac{h}{z}\,\phi(0) + \partial_w \phi(0)\right] = h\phi(0). \tag{41}$$

b) The Virasoro algebra is defined by the commutator

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12} m(m^2-1)\delta_{m,-n}, \qquad \forall\, m, n \in \mathbb{Z}. \tag{42}$$

We can use it to calculate the primary-state expectation value $\langle\phi|L_m L_{-m}|\phi\rangle$ by writing

$$\langle\phi|L_m L_{-m}|\phi\rangle = \langle\phi|[L_m, L_{-m}]|\phi\rangle. \tag{43}$$

To establish the truth of this will require most of the work in this exercise.

1. First, note that since $\phi(w)$ was introduced as a primary field of weight h, it too enjoys a Laurent series

$$\phi(w) = \sum_{n\in\mathbb{Z}} \phi_n\, w^{-n-h}. \tag{44}$$

We postulate that its action on the vacuum $\phi(w)|0\rangle$ should be non-singular at all times, but in particular at initial time $\tau_i = -\infty$ which corresponds to $w_i = e^{\frac{2\pi}{l}(\tau_i - i\sigma_i)} = 0$ on the complex plane.[2] For $\phi(w)|0\rangle$ not to blow up at $w = 0$ requires

$$\phi(0)|0\rangle = \sum_{n\in\mathbb{Z}} \lim_{w\to 0} \frac{1}{w^{n+h}}\,\phi_n|0\rangle < \infty \qquad \Leftrightarrow \qquad \phi_n|0\rangle = 0 \quad \forall\, n+h > 0. \tag{45}$$

Now, enforcing condition (45) while acting with $\phi(0)$ on the vacuum gives

$$\begin{aligned} |\phi\rangle = \phi(0)|0\rangle &= \sum_{n\in\mathbb{Z}} \lim_{w\to 0} \frac{1}{w^{n+h}}\,\phi_n|0\rangle \overset{(45)}{=} \sum_{n\le -h} \lim_{w\to 0} \frac{1}{w^{n+h}}\,\phi_n|0\rangle \\ &= \sum_{n<-h} \underbrace{\lim_{w\to 0} \frac{1}{w^{n+h}}}_{0}\,\phi_n|0\rangle + \frac{1}{w^{-h+h}}\,\phi_{-h}|0\rangle = \phi_{-h}|0\rangle. \end{aligned} \tag{46}$$

This is conformal field theory's famous operator-state correspondence.

2. Having worked this hard to find $|\phi\rangle = \phi_{-h}|0\rangle$, we want to use this equation to work out the effect of L_m on $|\phi\rangle$. We need one more tool for this, however, and that is $[L_m, \phi_n]$. Using the integral representation of the Laurent series coefficients,

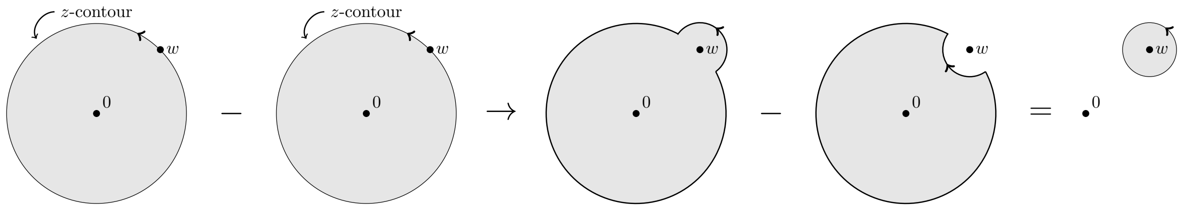
$$\phi_n = \oint_{C_0} \frac{\mathrm{d}w}{2\pi i}\, w^{n+h-1}\phi(w), \tag{47}$$

[2] See exercise 1.c) on assignment 7 for details concerning the conformal map from the cylinder to the plane.

we calculate

$$
\begin{aligned}
[L_m, \phi_n] &= \oint_{C_0} \frac{\mathrm{d}z}{2\pi i} z^{m+1}\, T(z) \oint_{C_0} \frac{\mathrm{d}w}{2\pi i} w^{n+h-1} \phi(w) - \oint_{C_0} \frac{\mathrm{d}w}{2\pi i} w^{n+h-1} \phi(w) \oint_{C_0} \frac{\mathrm{d}z}{2\pi i} z^{m+1}\, T(z) \\
&= \oint_{C_0} \frac{\mathrm{d}w}{2\pi i} \oint_{C_w} \frac{\mathrm{d}z}{2\pi i}\, z^{m+1}\, w^{n+h-1}\, \mathcal{R}[T(z)\, \phi(w)],
\end{aligned}
\tag{48}
$$

where we reemployed the contour deformation trick first used in exercise 2 on assignment 7.



By inserting the radially ordered OPE (36), we get

$$
\begin{aligned}
[L_m, \phi_n] &= \oint_{C_0} \frac{\mathrm{d}w}{2\pi i} \oint_{C_w} \frac{\mathrm{d}z}{2\pi i}\, z^{m+1}\, w^{n+h-1} \left[\frac{h\, \phi(w)}{(z-w)^2} + \frac{\partial_w \phi(w)}{z-w} \right] \\
&= \oint_{C_0} \frac{\mathrm{d}w}{2\pi i} \left[(m+1) w^m\, w^{n+h-1}\, h\, \phi(w) + w^{m+1}\, w^{n+h-1}\, \partial_w \phi(w) \right],
\end{aligned}
\tag{49}
$$

where we used the Cauchy-Riemann formula to execute the z-integration,

$$
\oint_{C_w} \frac{\mathrm{d}z}{2\pi i} \frac{f(z)}{(z-w)^n} = \frac{1}{(n-1)!} f^{(n-1)}(w). \tag{50}
$$

After partially integrating the last term in eq. (49),[3] we can reverse-apply eq. (47) to obtain

$$
\begin{aligned}
[L_m, \phi_n] &= \oint_{C_0} \frac{\mathrm{d}w}{2\pi i} \left[h\,(m+1) w^{m+n+h-1}\, \phi(w) - \underbrace{(\partial_w w^{m+n+h})}_{(m+n+h)\, w^{m+n+h-1}} \phi(w) \right] \\
&= [h\,(m+1) - (m+n+h)] \oint_{C_0} \frac{\mathrm{d}w}{2\pi i} \left[w^{m+n+h-1}\, \phi(w) \right] \\
&= [(h-1)\, m - n]\, \phi_{m+n}.
\end{aligned}
\tag{51}
$$

3. Finally, all is in place to tackle $L_m|\phi\rangle$.

$$
L_m|\phi\rangle = L_m \phi_{-h}|0\rangle = \phi_{-h} L_m|0\rangle + [L_m, \phi_{-h}]|0\rangle = \phi_{-h} L_m|0\rangle + [(h-1)\, m + h]\, \phi_{m-h}|0\rangle. \tag{52}
$$

By the same reasoning as employed in eq. (45), regularity of $T(z) = \sum_{m \in \mathbb{Z}} z^{-m-2} L_m$ at initial time $z = 0$ requires $L_m|0\rangle = 0\ \forall\, m > -2$. Since m was given as $m \in \mathbb{N}_0$ by the exercise, this takes care of the $\phi_{-h} L_m|0\rangle$ term in eq. (52). The second term is zero if $\phi_{m-h}|0\rangle = 0$, which by eq. (45) is the case for all $m + h - h = m > 0$. Thus we have

$$
L_m|\phi\rangle = 0 \qquad \forall\, m > 0. \tag{53}
$$

The case $m = 0$ was already treated in part a) where we found $L_0|\phi\rangle = h|\phi\rangle$. Unfortunately, we do not learn much from

$$
\langle \phi | L_0 L_0 | \phi \rangle = h^2 \langle \phi | \phi \rangle. \tag{54}
$$

But for all $m > 0$, thanks to eq. (53), we can now confirm our initial statement $\langle \phi | L_m L_{-m} | \phi \rangle = \langle \phi | [L_m, L_{-m}] | \phi \rangle$. Inserting the Virasoro commutator (42), we hence get

$$
\langle \phi | L_m L_{-m} | \phi \rangle \overset{(42)}{=} \langle \phi | 2m\, L_0 + \frac{c}{12} m(m^2-1) | \phi \rangle = 2mh\, \langle \phi | \phi \rangle + \frac{c}{12} m(m^2-1)\, \langle \phi | \phi \rangle. \tag{55}
$$

[3] Where we pick up no boundary terms due to integrating along the closed contour C_0.

For $m = 1$, eq. (55) becomes

$$\langle\phi|L_1 L_{-1}|\phi\rangle = 2h\,\langle\phi|\phi\rangle, \tag{56}$$

which demonstrates that $h \geq 0$ is a necessary condition for the absence of negative norm states. For $m \to \infty$, we get exactly the same constraint for the Virasoro algebra's central charge, $c \geq 0$.

c) Lastly, we show that in a unitary CFT, a primary field $|\phi^0\rangle$ of dimension $h = 0$ must be constant. Utilizing again the operator-state correspondence, this follows immediately from considering the norm of the state $\lim_{w\to 0} \partial_w \phi^0(w)|0\rangle$,

$$\lim_{w\to 0} ||\partial_w \phi^0(w)|0\rangle||^2 \overset{(40)}{=} \lim_{w\to 0} ||L_{-1}\phi^0(w)|0\rangle||^2 = ||L_{-1}|\phi^0\rangle||^2 = \langle\phi^0|L_1 L_{-1}|\phi^0\rangle \overset{(56)}{=} 2h\langle\phi^0|\phi^0\rangle = 0,$$

$$\Rightarrow \quad \partial_w \phi^0(w) = 0, \text{ i.e. } \phi^0(w) = \phi^0 \text{ is constant.} \tag{57}$$

3 Bonus question: Verma modules and descendant fields

The Verma module V_{h_j} associated with the primary field ϕ_j of dimension h_j was defined in the lecture as the span of states $|\phi_j^{\{k_i\}}\rangle = |\phi_j^{k_1,\dots,k_n}\rangle = L_{-k_1}\dots L_{-k_n}|\phi_j\rangle$. These states are created from the $PSL(2,\mathbb{C})$-invariant vacuum by insertion of the descendant field $\phi_j^{\{ki\}}(z)$ at $z = 0$, i.e.

$$|\phi_j^{\{ki\}}\rangle = \phi_j^{\{ki\}}(0)|0\rangle. \tag{58}$$

a) What is the L_0 eigenvalue of $L_{-k_1}\dots L_{-k_n}|\phi_j\rangle$?

b) Show that the descendant field $\phi_j^k(w)$ corresponding to the state $|\phi_j^k\rangle = L_{-k}|\phi_j\rangle$ is given by

$$\phi_j^k(w) = \oint_{C_w} \frac{\mathrm{d}z}{2\pi i}\,(z-w)^{1-k}\,T(z)\,\phi_j(w). \tag{59}$$

Hint: First perform a Taylor expansion for $T(z)$ about $z = w$ and then expand $T(z)$ into Virasoro modes.

a) The L_0 eigenvalue of a basis state $|\phi_j^{\{k_i\}}\rangle = L_{-k_1}\dots L_{-k_n}|\phi_j\rangle$ of the Verma module V_{h_j} can be computed with the help of the Virasoro algebra

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n}, \qquad \forall\, m,n \in \mathbb{Z}. \tag{60}$$

From it, we gather that all L_{-k} with $k > 0$ act as creation operators on eigenstates $L_0|\phi\rangle = h|\phi\rangle$ of L_0,

$$L_0 L_{-k}|\phi\rangle = L_{-k}L_0|\phi\rangle + [L_0, L_{-k}]|\phi\rangle = hL_{-k}|\phi\rangle + (0+k)L_{-k}|\phi\rangle = (h+k)L_{-k}|\phi\rangle. \tag{61}$$

By definition of the Verma module, the indices k_i of the Virasoro generators are ordered, i.e. $k_i \geq k_{i-1} > 0\ \forall\, i \in \{2,3,\dots,n\}$, and in particular, all larger than zero. For the state $|\phi_j^{\{k_i\}}\rangle$, we thus have

$$L_0|\phi_j^{\{k_i\}}\rangle = L_0 \prod_{i=1}^n L_{-k_i}|\phi_j\rangle = \Big(h_j + \sum_{i=1}^n k_i\Big)\prod_{i=1}^n L_{-k_i}|\phi_j\rangle = \Big(h_j + \sum_{i=1}^n k_i\Big)|\phi_j^{\{k_i\}}\rangle. \tag{62}$$

b) Descendant fields are a new class of central objects in conformal field theory. They are not primary but rather secondary operators and enter the theory as coefficients via the operator product expansion of primary fields with the energy-momentum tensor, i.e.

$$T(z)\,\phi_j(w) = \sum_{k=0}^{\infty}(z-w)^{k-2}\,\phi_j^k(w) = \frac{\phi_j^0(w)}{(z-w)^2} + \frac{\phi_j^1(w)}{z-w} + \dots \tag{63}$$

Note: Comparing this expansion with the OPE eq. (36) we wrote down earlier, we can immediately identify $\phi_j^0(w) = h_j\,\phi_j(w)$ and $\phi_j^1(w) = \partial_w \phi_j(w)$.

We can pick out the $\phi_j^k(w)$ term in the sum, by multiplying it with $(z-w)^{1-k}$ and integrating around the singularity at $z = w$:

$$\begin{aligned} \oint_{C_w} \frac{\mathrm{d}z}{2\pi i}\,(z-w)^{1-k}\,T(z)\,\phi_j(w) &= \oint_{C_w} \frac{\mathrm{d}z}{2\pi i}\,(z-w)^{1-k}\,\sum_{m=0}^{\infty}(z-w)^{m-2}\,\phi_j^m(w) \\ &= \sum_{m=0}^{\infty}\oint_{C_w}\frac{\mathrm{d}z}{2\pi i}\,\frac{\phi_j^m(w)}{(z-w)^{k-m+1}} = \phi_j^k(w) \end{aligned} \tag{64}$$

Note: Recalling

$$L_m = \oint_{C_0} \frac{\mathrm{d}z}{2\pi i} z^{m+1}\,T(z), \tag{65}$$

we can also very easily confirm that the descendant states are created from the primary ones by the action of the Virasoro generators,

$$|\phi_j^k\rangle \overset{(58)}{=} \phi_j^k(0)\,|0\rangle \overset{(64)}{=} \oint_{C_0} \frac{\mathrm{d}z}{2\pi i}\,z^{1-k}\,T(z)\,\phi_j(0)\,|0\rangle \overset{(65)}{=} L_{-k}\phi_j(0)\,|0\rangle = L_{-k}|\phi_j\rangle \tag{66}$$