Assignment 3

Due: Week beginning 04.05.2015.

Problem 3.1 (Wick's theorem reloaded):
In the lecture we have defined

$$\langle \phi(x_1)\ldots\phi(x_n)\rangle_0 = e^{\frac{1}{2}\frac{\delta}{\delta\phi}D_F\frac{\delta}{\delta\phi}}\phi(x_1)\ldots\phi(x_n)|_{\phi=0}. \tag{1}$$

Use this expression to prove Wick's theorem, i.e. show that

$$\langle \phi(x_1)\ldots\phi(x_{2n})\rangle_0 = D_F(x_1-x_2)D_F(x_3-x_4)\ldots D_F(x_{2n-1}-x_{2n}) + \text{ all other contractions.}$$

Hint: First show that

$$\langle \phi(x_1)\ldots\phi(x_{2n})\rangle_0 = \frac{1}{2^n n!}\sum_\sigma D_F(x_{\sigma_1}-x_{\sigma_2})D_F(x_{\sigma_3}-x_{\sigma_4})\cdots D_F(x_{\sigma_{2n-1}}-x_{\sigma_{2n}}), \tag{2}$$

where the sum is over all permutations σ of $2n$ elements, and then bring this into the required form.

Problem 3.2 (Two-point function and functional determinant for a free theory):
In this exercise we will evaluate the two-point function

$$\langle\Omega|\, T\phi(x_1)\phi(x_2)\,|\Omega\rangle = \lim_{T\to\infty}\frac{\int \mathcal{D}\phi\,\phi(x_1)\phi(x_2)\,e^{i\,S[\phi]}}{\int\mathcal{D}\phi\,e^{i\,S[\phi]}} \tag{3}$$

for the free real scalar field theory,

$$S[\phi] = \int d^4x\,\mathcal{L} = \int d^4x\,\tfrac{1}{2}\left(\partial_\mu\phi\partial^\mu\phi - (m^2-i\epsilon)\,\phi^2\right)\,, \tag{4}$$

by means of path integral techniques.

Note: We are taking real time in the path integral, but have shifted $m^2\to m^2-i\epsilon$ as this can be shown to have the same effect as taking $t\to t(1-i\epsilon)$.

To evaluate the path integral we first introduce a UV- and an IR-cutoff, i.e. we discretise and restrict to finite volume. Therefore, we get

$$\mathcal{D}\phi = \prod_i d\phi(x_i)\,,$$
$$\phi(x_i) = \frac{1}{V}\sum_n e^{-ik_n\cdot x_i}\phi(k_n)\,,$$

where $k_n^\mu = \frac{2\pi n^\mu}{L}$ with $n^\mu\in\mathbb{Z}$, $|k^\mu|<\frac{\pi}{a}$ and $V=L^4$.[1] L and a are the order parameter of the volume and the lattice spacing, respectively.

[1]Note that due to the cutoff, although not explicitly indicated, the sums over the wavenumbers k_n^μ have finitely many summands.

a) Why is the measure, when we go to Fourier space, given by

$$\mathcal{D}\phi(x) = J \prod_{k_n^0 > 0} d\Re\phi(k_n) d\Im\phi(k_n)\,, \tag{5}$$

with some constant J, and not by (5) with k_n^0 unconstrained?

b) By proceeding as in Problem 2.2 d) and e), show that

$$\int \mathcal{D}\phi e^{i\,S[\phi]} = J \prod_{k_n^0} \sqrt{\frac{-i\,\pi\,V}{m^2 - i\epsilon - k_n^2}} \sqrt{\frac{-i\,\pi\,V}{m^2 - i\epsilon - k_n^2}} = J \prod_{\text{all } k_n} \sqrt{\frac{-i\,\pi\,V}{m^2 - i\epsilon - k_n^2}}\,. \tag{6}$$

Hint: Use (or show) that for $\Re(b) > 0$, the Gaussian integral $\int_{-\infty}^{\infty} dx\, e^{-b\,x^2}$ is well-defined and gives $\sqrt{\frac{\pi}{b}}$. Argue why this theorem is applicable here.

c) Before we continue with the two-point function, we want to relate (6) with the *functional determinant*. To do so, convince yourself that

$$\left(\prod_k \int d\xi_k\right) e^{-\xi_i B^{ij} \xi_j} = \prod_i \sqrt{\frac{\pi}{b^i}} = \text{const} \times \frac{1}{\sqrt{\det B}} \tag{7}$$

for B some symmetric positive definite $N \times N$ matrix (more generally it suffices that the eigenvalues have a positive real part) and $\boldsymbol{\xi} \in \mathbb{R}^N$.

Hint: Do an integral transformation to go to the diagonal space of B.

d) Now rewrite the action (4) into the form

$$S[\phi] = \phi \cdot D \cdot \phi + \text{surface terms}$$

with D some differential operator and argue in analogy to c) that

$$\int \mathcal{D}\phi\, e^{i\,S[\phi]} = \text{const}\, \frac{1}{\sqrt{\det D}}\,.$$

Further, give the differential operator D.

Hint: You can neglect the surface terms in your argument because we are assuming natural boundary conditions.

e) After this short interlude, we want to calculate the numerator on the right hand side of (3). Therefore, use the same regularisation as for the denominator and go again to Fourier space. Then integrate out the Fourier coefficients. You should find the following expression for the numerator:

$$\int \mathcal{D}\phi\, \phi(x_1)\phi(x_2)\, e^{i\,S[\phi]} = \frac{J}{V^2} \sum_m e^{-ik_m\cdot(x_1 - x_2)} \frac{-i\,V}{m^2 - i\epsilon - k_m^2} \left(\prod_{k_n^0 > 0} \frac{-i\,\pi\,V}{m^2 - i\epsilon - k_n^2}\right)\,. \tag{8}$$

Now, take the ratio of (8) and (6) and explain why it actually should be

$$\int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2 + i\,\varepsilon} e^{-ik\cdot(x_1 - x_2)}\,.$$

Hint: To evaluate this path integral you will need also the higher momenta of the Gaussian integral.

Remark: This exercise shows that the path integral and all correlators are really defined in the IR + UV regularized theory, with the continuum limit taken in the end. Explain in particular why it poses no problem here that the "functional determinant" itself becomes divergent in this limit.