Quantum Field Theory II - Final Exam

Tillmann Plehn, Heidelberg University

28.07.2012, 10:00 - 12:00

Problem 1: One-loop calculations in QED

Let us consider QED with a massless fermion field, which we describe by means of the Lagrangian $\mathcal{L} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + i\bar{\psi}\not{D}\psi$, with the standard notation for the gauge covariant derivative $D_\mu = \partial_\mu + ieA_\mu$.

1. Using the corresponding Feynman rules, work out the general expression for the one-loop correction to the fermion propagator, $i\mathcal{M}[f(p) \to f(p)]$. Show explicitly that the result can be cast into $i\mathcal{M}[f(p) \to f(p)] = 8\pi\alpha_{\text{em}}\bar{u}(p)\not{p}F(p^2)u(p)$, where $\alpha_{\text{em}} = e^2/4\pi$ and the one-loop form factor $F(p^2)$ reads
$$F(p^2) = -\int_0^1 \mathrm{d}x \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{x}{[l^2 + x(1-x)p^2]^2}. \tag{1}$$
Bear in mind that the Feynman parameters are a suitable handle for computing one-loop integrals. In particular, the following identity might be useful,
$$\frac{1}{a^p b^q} = \frac{\Gamma(p+q)}{\Gamma(p)\Gamma(q)} \int_0^1 \mathrm{d}x \, \frac{x^{p-1}(1-x)^{q-1}}{[ax + b(1-x)]^{p+q}}. \tag{2}$$
Remember also that $\int \mathrm{d}^4 k \, k = 0$.

2. By similar arguments, one can prove that the one-loop correction to the photon propagator reads $i\mathcal{M}[\gamma(p) \to \gamma(p)] = \epsilon^*_\mu(p)\Pi^{\mu\nu}(p^2)\epsilon_\nu(p)$, with the polarization tensor yielding
$$\Pi^{\mu\nu}(p^2) = \alpha_{\text{em}} \left[F_1(p^2) g^{\mu\nu} + F_2(p^2)\frac{p^\mu p^\nu}{p^2} \right]. \tag{3}$$
Upon explicit calculation, one finds that $F_1(p^2) = -F_2(p^2)$. Explain why this is indeed a necessary requirement for QED to be consistently defined as a gauge theory. What is the physical interpretation of this result?

Problem 2: Renormalization Group Equation for QED

Let us consider the renormalization of QED at one-loop. To fix the notation, we first set along the relation between the bare coupling constant e_0, the fermion field ψ_0, the fermion mass m_0, and the photon field A_0; and their corresponding renormalized quantities:

$$e_0 = Z_e e, \qquad \psi_0 = Z_\psi^{1/2}\psi, \qquad m_0 = Z_m m, \qquad (A_0)_\mu = Z_A^{1/2} A_\mu. \tag{4}$$

The renormalization of the coupling constant we can fix by studying the fermion-photon interaction,

$$\mathcal{L} \supset e_0 \bar{\psi}_0 \gamma^\mu \psi_0 (A_0)_\mu = eZ\bar{\psi}\gamma^\mu\psi A_\mu \tag{5}$$

while the renormalization of the fermion mass ensues from the mass term in the Lagrangian,

$$\mathcal{L} \supset m_0 \bar{\psi}_0 \psi_0 = Z_m m \bar{\psi}\psi. \tag{6}$$

Finally, we need the renormalization factors, which in the $\overline{\text{MS}}$ scheme render

$$
\begin{aligned}
Z^{\overline{\text{MS}}} &= 1 - \frac{\alpha}{2\pi}\frac{1}{\epsilon}, \qquad & Z_{\psi}^{\overline{\text{MS}}} &= 1 - \frac{\alpha}{2\pi}\frac{1}{\epsilon}, \\
Z_{A}^{\overline{\text{MS}}} &= 1 - \frac{2\alpha}{3\pi}\frac{1}{\epsilon}, \qquad & Z_{m}^{\overline{\text{MS}}} &= 1 - \frac{2\alpha}{\pi}\frac{1}{\epsilon}.
\end{aligned} \tag{7}
$$

From eqs. (4) and (5), one gets the following relations:

$$
\text{i) } \alpha_0 = \alpha Z_A^{-1} Z_{\psi}^{-2} Z^2 \mu^{\epsilon}, \qquad \text{ii) } m_0 = m Z_m Z_{\psi}^{-1} \mu^{\epsilon}. \tag{8}
$$

1. From the logarithms of eq. (8) i) and ii), derive the β-function at $\mathcal{O}(\alpha^2)$ and the mass anomalous dimension γ_m at $\mathcal{O}(\alpha)$. Recall that these quantities are defined as $\beta \equiv \frac{\mathrm{d}\alpha(\mu)}{\mathrm{d}\ln(\mu)}$ and $\gamma_m \equiv \frac{1}{m}\frac{\mathrm{d}m(\mu)}{\mathrm{d}\ln(\mu)}$.

   Hint: Bear in mind that $\frac{\mathrm{d}\alpha_0}{\mathrm{d}\ln(\mu)} = 0$ and $\frac{\mathrm{d}m_0}{\mathrm{d}\ln(\mu)} = 0$.

2. Use the above results to solve the equations $\beta \equiv \frac{\mathrm{d}\alpha(\mu)}{\mathrm{d}\ln(\mu)}$ and $\gamma_m \equiv \frac{1}{m}\frac{\mathrm{d}m(\mu)}{\mathrm{d}\ln(\mu)}$ explicitly. Sketch the resulting scale dependence of the renormalized coupling constant $\alpha(\mu)$ and the fermion mass $m(\mu)$.

3. Suppose that we modify QED in such a way that the new β-function of the resulting theory can now be written as $\beta = \beta_{\text{QED}}\alpha^2 + \beta_4\alpha^4$, with β_4 being a real parameter. Discuss the value and the stability of the corresponding fixed points for the possible values of β_4. In which cases would the theory be infrared safe? And asymptotically free?

Problem 3: QCD Feynman rules and color factors

Consider the Feynman diagrams displayed in fig. 1, which describe part of the tree-level and one-loop contributions, respectively, to the production of top-quark pairs at the LHC.

1. For each of these diagrams, write down the complete scattering matrix element $i\mathcal{M}(u\bar{u} \to t\bar{t})$, making use the QCD Feynman rules.

2. Compute (just) the overall color actor for

   (a) the squared of the tree-level diagram.

   (b) the interference of the tree-level and the one-loop diagrams: $M_{\text{tree}}^{\dagger} M_{\text{loop}}$.

   Useful Identities:

$$
\begin{aligned}
\text{Tr}(T^A T^B) &= \frac{1}{2}\delta^{AB}, \qquad & (T^A)_{ij}(T^A)_{kl} &= \frac{1}{2}\left(\delta_{il}\delta_{jk} - \frac{1}{N_{\text{c}}}\delta_{ij}\delta_{kl}\right), \\
f^{ABC} f^{ABD} &= 3\delta^{CD}, \qquad & 2T^B T^C &= [T^B, T^C] + T^B, T^C.
\end{aligned} \tag{9}
$$

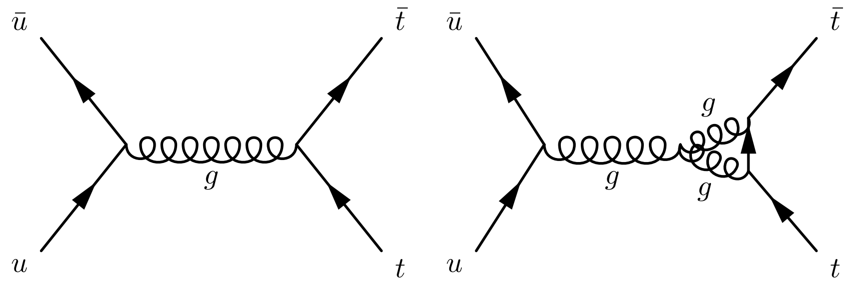

Figure 1: Tree-level and one-loop contributions to the production of top-quark pairs.