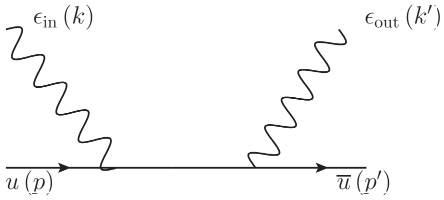
Quantum Field Theory II
Lecturer: Jürgen Berges
Tutor: Viraf Mehta

Institut für Theoretische Physik
Universität Heidelberg
Sommersemester 2014/15

Assignment 1

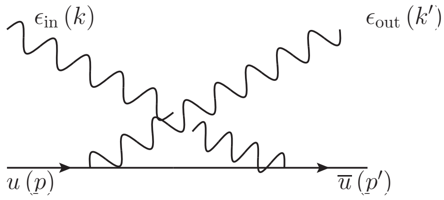
Due: Week beginning 20.04.2015.

Problem 1.1 (Compton Scattering):
Consider the Compton scattering process $e^-\gamma \to e^-\gamma$ in QED. Use the Feyman rules on page 4 to derive the amplitude for the tree level diagram,



$$= i\mathcal{M}_1 = i(-ie)^2\overline{u}_r(\vec{p}')\not{\epsilon}_{\text{out}}\frac{(\not{p}+\not{k}+m)}{(p+k)^2-m^2}\not{\epsilon}_{\text{in}}u_s(\vec{p}). \qquad (1)$$

Also compute the contribution from

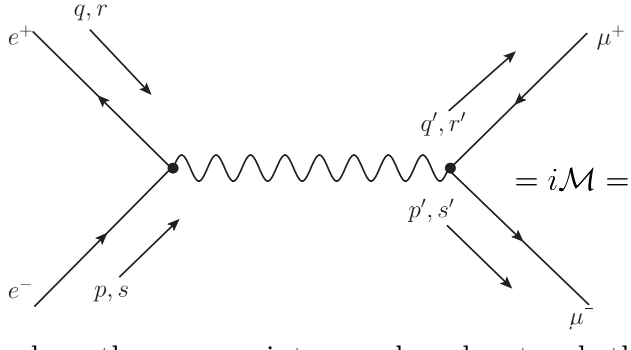


The total amplitude, at order e^2, is the sum of these two diagrams. Show that, if ϵ_{in} is replaced by the incoming photon momentum k, then the *total* amplitude vanishes. Check that the same holds true if ϵ_{out} is replaced by k'.

Hint: You may find $(\not{p}-m)u(\vec{p})=0$ useful.

Problem 1.2 ($e^-e^+ \to \mu^-\mu^+$ Scattering Amplitude):
For this question, use the fact that a muon, $\mu^\pm$, is a Dirac fermion with mass $m_\mu \gg m_e$ and satisfies the same Feynman rules as the electron.

a.) Using the Feynman rules for QED on page 4, show that the amplitude for $e^-e^+ \to \mu^-\mu^+$ is given, at lowest order in e, by,



$$= i\mathcal{M} = -i(-ie)^2\frac{[\overline{v}^e_r(\vec{q})\gamma_\mu u^e_s(\vec{p})][\overline{u}^m_{s'}(\vec{p}')\gamma^\mu v^m_{r'}(\vec{q}')]}{(p+q)^2+i\varepsilon} \qquad (2)$$

where the superscripts e and m denote whether the spinors satisfy the Dirac equation for the electrons or muons. Briefly comment as to why this is the only contributing diagram *unlike* for $e^-e^+ \to e^-e^+$ scattering.

b.) Prove the following identities:

i.) $\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4\eta^{\mu\nu}$

ii.) $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\sigma) = 0$

iii.) $\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho) = 4(\eta^{\mu\nu}\eta^{\sigma\rho} - \eta^{\mu\sigma}\eta^{\nu\rho} + \eta^{\mu\rho}\eta^{\nu\sigma})$

iv.) $\sum_{s,s'}[\overline{v}_{s'}(p')\gamma^\nu u_s(p)]^*[\overline{v}_{s'}(p')\gamma^\mu u_s(p)] = 4[p^\nu p'^\mu + p^\mu p'^\nu - (p.p' + m^2)\eta^{\mu\nu}]$

Hint: You may find the following relations useful,

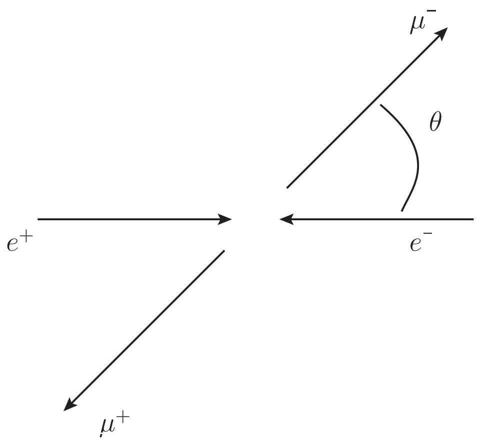
$$\begin{aligned}\sum_s u_s(\vec{p})\overline{u}_s(\vec{p}) &= \gamma.p + m,\\ \sum_s v_s(\vec{p})\overline{v}_s(\vec{p}) &= \gamma.p - m.\end{aligned} \tag{3}$$

c.) Let m, M denote the electron and muon masses, respectively. Show that

$$\sum_{srs'r'} |\mathcal{M}|^2 = \frac{e^4}{s^2}\mathrm{Tr}[(\gamma.p' + M)\gamma^\mu(\gamma.q' - M)\gamma^\nu]\mathrm{Tr}[(\gamma.p + m)\gamma_\nu(\gamma.q - m)\gamma_\mu] \tag{4}$$

where $s = (p+q)^2$.

d.) This can be simplified further, assuming that the momentum components are sufficiently large enough and thus, one can neglect the electron and muon masses as a good approximation.



In the centre-of-mass frame,

$$\vec{q} = -\vec{p}; \qquad \vec{q}' = -\vec{p}'; \tag{5}$$

$$\text{and} \qquad q^0 = p^0 = |\vec{p}|; \qquad q'^0 = p'^0 = |\vec{p}'| \tag{6}$$

by setting $m = M = 0$. Show that

$$\sum_{srs'r'} |\mathcal{M}|^2 = \frac{32e^4}{s^2}[p.p'q.q' + p.q'q.p'] = 4e^4\left(1 + \cos^2\theta\right) \tag{7}$$

where θ is the scattering angle in the centre-of-mass frame.

Problem 1.3 (**Symmetries of classical electrodynamics**):
We consider **classical** massless electrodynamics,

$$\mathcal{L}_{\mathrm{ED}} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \overline{\psi}\gamma^\mu(i\partial_\mu + A_\mu)\psi \tag{8}$$

where the field strength is defined as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. This theory is invariant under local $U(1)$ gauge transformations, with the fields transforming as,

$$\begin{aligned}\psi &\longrightarrow e^{i\alpha(x)}\psi,\\ A_\mu &\longrightarrow A_\mu + \partial_\mu\alpha(x).\end{aligned} \tag{9}$$

a.) Find the equations of motion for the gauge and fermion fields.

b.) The energy-momentum tensor, T^μ_ν, is the conserved Noether current associated to the space-time translations

$$x^\mu \longrightarrow x'^\mu = x^\mu - a^\nu \delta^\mu_\nu, \tag{10}$$

with

$$\begin{aligned} A_\mu(x) &\longrightarrow A'_\mu(x') = A_\mu(x), \\ \psi(x) &\longrightarrow \psi'(x') = \psi(x). \end{aligned} \tag{11}$$

i.) Show that, for classical electrodynamics, it is given by

$$T^\mu_\nu = -F^{\mu\rho}\partial_\nu A_\rho + \delta^\mu_\nu \frac{1}{4} F^{\rho\sigma}F_{\rho\sigma} + i\overline{\psi}\gamma^\mu \partial_\nu \psi. \tag{12}$$

Hint: Noether's theorem states that given a symmetry transformation parameterised by[1] ε^a inducing the infinitesimal transformations

$$\begin{aligned} x^\mu &\longrightarrow x'^\mu = x^\mu - \varepsilon^a \mathcal{E}^\mu_a + \mathcal{O}(\varepsilon^2), \\ \chi_i(x) &\longrightarrow \chi'_i(x) = \chi_i(x) + \varepsilon^a \Delta_{ai} + \mathcal{O}(\varepsilon^2), \end{aligned} \tag{13}$$

where χ_i is any of the fields, there exists a conserved current, J^μ_a, given by:

$$J^\mu_a = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \chi_i)} \Delta_{ai} - \mathcal{E}^\mu_a \mathcal{L}. \tag{14}$$

ii.) Show that T^μ_ν is not gauge invariant.

c.) We may restore gauge invariance by "improving" the energy-momentum tensor. A conserved current, J^μ_a, can always be improved with the help of a (non-conserved) antisymmetric tensor, $L^{\mu\nu}_a = -L^{\nu\mu}_a$.

i.) Show that the improved current

$$\tilde{J}^\mu_a = J^\mu_a + \partial_\nu L^{\mu\nu}_a \tag{15}$$

is also conserved and gives rise to the same conserved charge, $\tilde{Q}_a$, as that of J^μ_a.
Hint: Recall that $Q_a = \int d^3x J^0_a$.

ii.) Now, focussing on the limit $\psi, \overline{\psi} \to 0$ for simplicity, use the equations of motion to find the antisymmetric tensor $L^{\rho\mu}_\nu$ that improves the energy-momentum tensor, restoring gauge invariance, *i.e.*

$$\Theta^\mu_\nu \big|_{\psi=0=\overline{\psi}} = (T^\mu_\nu + \partial_\rho L^{\mu\rho}_\nu)\big|_{\psi=0=\overline{\psi}} = -F^{\mu\rho}F_{\nu\rho} + \delta^\mu_\nu \frac{1}{4} F^{\rho\sigma}F_{\rho\sigma}. \tag{16}$$

d.) Finally, massless electrodynamics has one additional symmetry: spacetime symmetries are enhanced with *scale invariance*:

$$\begin{aligned} x^\mu &\longrightarrow x'^\mu = e^{-\beta} x^\mu; \\ A_\mu(x) &\longrightarrow A'_\mu(x') = e^{\beta} A_\mu(x); \qquad\qquad \beta \in \mathbb{R} \\ \psi(x) &\longrightarrow \psi'(x') = e^{\frac{3\beta}{2}} \psi(x); \end{aligned} \tag{17}$$

in addition to the usual Poincaré invariance.

i.) Show that the action $S = \int d^4x \, \mathcal{L}_{\text{ED}}$ is invariant under scale transformations.

ii.) Show that the associated current, S^μ, can be written as:

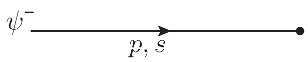
$$S^\mu = x^\nu T^\mu_\nu + U^\mu, \tag{18}$$

and give the expression of U^μ.

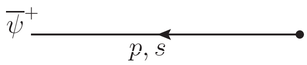
[1] a runs over the number of independent transformations, *e.g.* $a = 1$ for a $U(1)$ symmetry or $a = \nu = 0, \ldots, 3$ for spacetime translations

The Feynman Rules for QED

Incoming fermions:

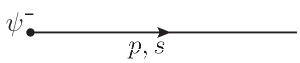


$u_s^{\psi}(p)$

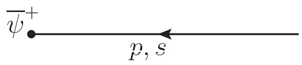


$\overline{v}_s^{\overline{\psi}}(p)$

Outgoing fermions:

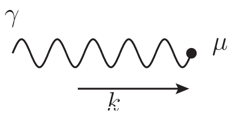


$\overline{u}_s^{\psi}(p)$



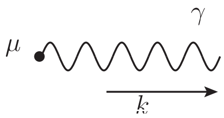
$v_s^{\overline{\psi}}(p)$

Incoming photon:



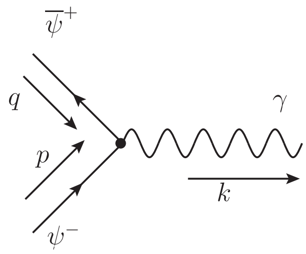
$\epsilon^{\mu}_{\text{in}}(k)$

Outgoing photon:



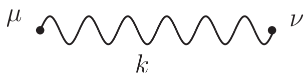
$\epsilon^{\mu}_{\text{out}}(k)$

Vertices:



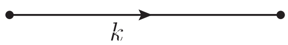
$-ie\gamma^{\mu}$

Photon propagator:



$$\frac{-i\eta_{\mu\nu}}{k^2+i\varepsilon}$$

Fermion propagator:



$$\frac{i\left(\not{k}+m_{\psi}\right)}{k^2-m_{\psi}^2+i\varepsilon}$$

Table 1: ψ and $\overline{\psi}$ are fermions and antifermions, *e.g.* the electron and positron respectively, and their electric charge is indicated.