05.07.2015

Problem 7.1 (One-loop structure of ϕ^4-theory)

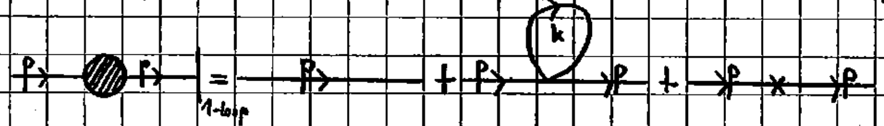
a) Renormalization of the scalar propagator  $= \dfrac{i}{p^2 - m^2 - M^2(p^2)}$ up to one-loop order:

Renormalization condition:  $\Big|_{p^2=m^2} = \dfrac{i}{p^2-m^2}$

This requirement is equivalent to

i) $M^2(p^2)\Big|_{p^2=m^2} = 0$, ii) $\dfrac{d}{dp^2} M^2(p^2)\Big|_{p^2=m^2} = 0.$

At one loop order,  receives contributions from



$$= \frac{i}{p^2-m^2} + \underset{\text{symmetry factor}}{\frac{-i\lambda}{2}} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2} + i(p^2\delta_z - \delta_m).$$

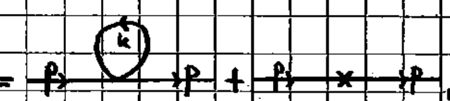
This follows from the Feynman rules obtained by splitting up the $\lambda\phi^4$-theory Langrangian like so

$$\mathcal{L}_{\lambda\phi^4} = \frac{1}{2}(\partial\phi_0)^2 - \frac{1}{2}m_0\phi_0 - \frac{\lambda_0}{4!}\phi_0^4$$

$$= \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m\phi^2 - \frac{\lambda}{4!}\phi^4 + \frac{1}{2}\delta_z(\partial\phi^2) - \frac{1}{2}\delta_m\phi^2 - \frac{\delta_\lambda}{4!}\phi^4,$$

where $\phi = Z^{-\frac{1}{2}}\phi_0$, $\delta_z = Z-1$, $\delta_m = Zm_0^2 - m$, $\delta_\lambda = Z^2\lambda_0 - \lambda$.

We know that

$$-iM^2(p^2)\Big|_{1\text{-loop}} = (1PI)\Big|_{1\text{-loop}} =$$ ,

thus i) and ii) imply

$$\frac{d}{dp^2} M^2(p^2)\Big|_{p^2=m^2} = -\delta_z = 0$$

$$, \quad M^2(p^2)\Big|_{p^2=m^2} = \frac{\lambda}{2}\int\frac{d^4k}{(2\pi)^4}\frac{i}{k^2-m^2} + \delta_m = 0$$

Thus to complete the renormalization of the propagator of $\lambda\phi^4$-theory at one-loop order, we simply need to calculate

$$\delta_m = -\frac{\lambda}{2}\int \frac{d^dk}{(2\pi)^d}\frac{i}{k^2-m^2} \overset{k^0=ik_E^0}{=} -\frac{\lambda}{2}\int\frac{d^dk_E}{(2\pi)^d}\frac{1}{k_E^2+m^2} = -\frac{\lambda}{2}\int\frac{d\Omega_d}{(2\pi)^d}\int dk_E \frac{k_E^{d-1}}{k_E^2+m^2},$$

where $\int d\Omega_d = \frac{2\pi^{\frac{d}{2}}}{\Gamma\left(\frac{d}{2}\right)}$. We substitute $x = \frac{m^2}{k_E^2+m^2}$, $dx = -\frac{2m^2k_E}{(k_E^2+m^2)^2}dk_E$,

$$\delta_m = -\frac{2\pi^{\frac{d}{2}}\lambda}{2(2\pi)^d\,\Gamma\left(\frac{d}{2}\right)}\int\left(-\frac{(k_E^2+m^2)^2}{2m^2k_E}dx\right)\frac{k_E^{d-1}}{k_E^2+m^2}, \qquad k_E = m\left(\frac{1}{x}-1\right)^{\frac{1}{2}}$$

$$= \frac{\lambda}{2(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\int dx\,\frac{1}{x}k_E^{d-2} = \frac{m^{d-2}\lambda}{2(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\int dx\,\underbrace{\frac{1}{x}\left(\frac{1}{x}-1\right)^{\frac{d}{2}-1}}_{x^{-\frac{d}{2}}(1-x)^{\frac{d}{2}-1}}$$

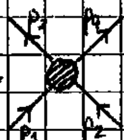
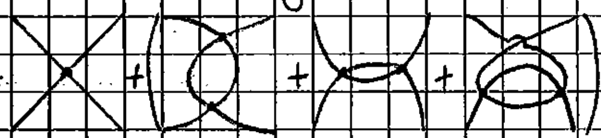
$$= \frac{m^{d-2}\lambda}{2(4\pi)^{\frac{d}{2}}\Gamma\left(\frac{d}{2}\right)}\frac{\Gamma\left(1-\frac{d}{2}\right)\Gamma\left(\frac{d}{2}\right)}{\Gamma\left(1-\frac{d}{2}+\frac{d}{2}\right)} = \frac{\lambda m^{d-2}}{2(4\pi)^{\frac{d}{2}}}\underbrace{\Gamma\left(1-\frac{d}{2}\right)}_{\frac{\Gamma(2-d/2)}{1-d/2}}$$

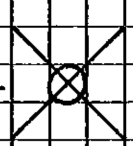
$$= \frac{\lambda m^{2-\frac{\epsilon}{2}}}{2(4\pi)^{2-\frac{\epsilon}{2}}}\frac{\Gamma\left(\frac{\epsilon}{2}\right)}{-\left(1-\frac{\epsilon}{2}\right)} = -\frac{\lambda m^{2-\frac{\epsilon}{2}}}{2(4\pi)^{2-\frac{\epsilon}{2}}}\left(1+\frac{\epsilon}{2}+\mathcal{O}(\epsilon^2)\right)\left(\frac{2}{\epsilon}-\gamma+\mathcal{O}(\epsilon)\right)$$

$$\xrightarrow{\epsilon\to 0} -\frac{m^2\lambda}{2(4\pi)^2}\left(\frac{2}{\epsilon}-\gamma+1\right) = -\frac{m^2\lambda}{32\pi^2}\left(\frac{2}{\epsilon}-\gamma+1\right)$$

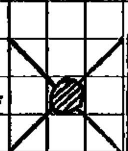
In the end, we found $M^2(p^2)=0\ \forall p$, at to least to one-loop order. This is a surprising result and a specialty of ϕ^4-theory in $d=4$ dimensions.

b) Renormalization of the two-particle scattering amplitude at one-loop

$iM(p_1p_2\to p_3p_4) =$  $=$ 

$+$  $= -i\lambda + (-i\lambda)^2(iV(s)+iV(t)+iV(s)) - i\delta_\lambda$

Our renormalization condition is

$iM(p_1p_2\to p_3p_4)\big|_{s=4m^2,t=u=0} =$  $\Big|_{s=4m^2,t=u=0} = -i\lambda.$

This amounts to $\delta_\lambda = (-i\lambda)^2\left(V(4m^2)+2V(0)\right)$, so all we need to

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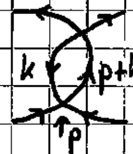
here is calculate $V(p^2)$, whose structure is irrespective of whether the diagram is s-, t-, or u-channel.



$$= \frac{1}{2}(-i\lambda)^2 \int \frac{d^dk}{(2\pi)^d} \frac{i}{k^2-m^2}\,\frac{i}{(p+k)^2-m^2} = (-i\lambda)^2\, i\, V(p^2)$$

$$V(p^2) = \frac{i}{2}\int \frac{d^dk}{(2\pi)^d}\frac{1}{k^2-m^2}\,\frac{1}{(p+k)^2-m^2} = \frac{i}{2}\int_0^1 dx \int \frac{d^dk}{(2\pi)^d}\frac{1}{\left(x((p+k)^2-m^2)+(1-x)(k^2-m^2)\right)^2}$$

$$x((p+k)^2-m^2)+(1-x)(k^2-m^2) = xp^2+2xpk+x(k^2-m^2)+k^2-m^2-x(k^2-m^2)$$

$$= k^2+2xpk+x^2p^2-x^2p^2+xp^2-m^2 = (\underbrace{k+xp}_{l})^2+\underbrace{x(1-x)p^2-m^2}_{\Delta^2}$$

$$V(p^2)=\frac{i}{2}\int_0^1 dx\int\frac{d^dl}{(2\pi)^d}\frac{1}{(l^2+\Delta^2)^2} = -\frac{1}{2}\int_0^1 dx\int\frac{d^dl_E}{(2\pi)^d}\frac{1}{(l_E^2-\Delta^2)^2} = -\frac12\int_0^1 dx\int\frac{d\Omega_d}{(2\pi)^d}\int_0^\infty dl_E\, \frac{l_E^{d-1}}{(l_E^2-\Delta^2)^2}$$

$$= -\frac12\int_0^1 dx\, \frac{2\pi^{\frac d2}}{(2\pi)^d\,\Gamma(\frac d2)}\int_1^0 \frac{(l_E^2-\Delta^2)^2}{-2\Delta^2 l_E}\,\frac{l_E^{d-1}}{(l_E^2-\Delta^2)^2}\,dz, \quad z=\frac{\Delta^2}{l_E^2-\Delta^2},\ l_E=\Delta\left(\frac1z-1\right)^{\frac12},\ dz=\frac{-2\Delta^2 l_E}{(l_E^2-\Delta^2)^2}\,dl_E$$

$$= \frac{-1}{2(4\pi)^{\frac d2}\,\Gamma(\frac d2)}\int_0^1 dx\, \underbrace{\int_0^1 dz\, z^{1-\frac d2}(1-z)^{\frac d2-1}}_{\Gamma(2-\frac d2)\Gamma(\frac d2)/\Gamma(2-\frac d2+\frac d2)}\,\Delta^{d-4} = -\frac{\Gamma(2-\frac d2)}{2(4\pi)^{\frac d2}}\int_0^1 dx\,\Delta^{d-4}$$

Using $\Gamma(2-\frac d2)=\Gamma(\frac\epsilon2)=\frac2\epsilon-\gamma+\mathcal{O}(\epsilon)$, $x^{-\frac\epsilon2}=e^{\ln x^{-\frac\epsilon2}}=e^{-\frac\epsilon2\ln x}=1-\frac\epsilon2\ln x+\mathcal{O}(\epsilon^2)$

we get

$$V(p^2) = -\frac{\Gamma(\frac\epsilon2)}{2(4\pi)^2}(4\pi)^{\frac\epsilon2}\int_0^1 dx\,\left(x(1-x)p^2-m^2\right)^{-\frac\epsilon2}$$

$$= -\frac{1}{32\pi^2}\left(\frac2\epsilon-\gamma+\mathcal{O}(\epsilon)\right)\left(1+\frac\epsilon2\ln(4\pi)+\mathcal{O}(\epsilon^2)\right)\int_0^1 dx\left[1-\frac\epsilon2\left(x(1-x)p^2-m^2\right)+\mathcal{O}(\epsilon^2)\right]$$

$$= -\frac{1}{32\pi^2}\int_0^1 dx\left[\frac2\epsilon-\gamma+\ln(4\pi)-\ln\left(x(1-x)p^2-m^2\right)\ \mathcal{O}(\epsilon)\right]$$

So the divergent counterterm δ_λ is given by

$$\delta_\lambda = (-i\lambda)^2\left(V(4m^2)+2V(0)\right) = \frac{\lambda^2}{32\pi^2}\int_0^1 dx\left[\frac6\epsilon-3\gamma+3\ln(4\pi)-\ln\left(x(1-x)4m^2-m^2\right)-2\ln(m^2)\right]$$

and for the two-particle scattering amplitude, we obtain the lengthy but finite result of

$$i\mathcal{M}(p_1 p_2 \to p_3 p_4) = -i\lambda - \frac{i\lambda}{32\pi^2}\int_0^1 dx \left[\ln\left(\frac{x(1-x)s - m^2}{x(1-x)4m^2 - m^2}\right) + \ln\left(\frac{x(1-x)t - m^2}{m^2}\right) + \ln\left(\frac{x(1-x)u - m^2}{m^2}\right)\right]$$

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Problem 8.1 (Superficial degree of divergence)

Aim: derivation of superficial degree of divergence D of QED

Notation:

- $E_{f/\gamma}$: number of external fermion / photon lines
- $P_{f/\gamma}$: number of internal fermion / photon lines (propagators)
- V : number of vertices
- L : number of loops

In $d=4$ dimensions, every loop gives a four-momentum integral. Every fermionic / photonic propagator gives one / two inverse powers of momentum. Thus,

$$D = 4L - P_f - 2P_\gamma.$$

Every propagator connects to two vertices while every external line connects to one vertex, since we are concerned solely with fully connected amputated diagrams, i.e. no external lines simply going through and no propagators on external lines.

Further every vertex connects two fermionic lines to one photon, so

$$V = \frac{1}{2}(E_f + 2P_f) = E_\gamma + 2P_\gamma,$$

and each propagator yields an integral over its undetermined momentum while every vertex gives a delta distribution, determining one momentum. Of these delta distr. however, one is used up to assure overall momentum conservation. So the number of momentum integrals (equals number of loops) is

$$L = P_f + P_\gamma - (V-1) = P_f + P_\gamma - V + 1.$$

Putting everything together, we can construct an expr.

for D which depends only on the external lines

$$D = 4L - P_f - 2P_\gamma = 4P_f + 4P_\gamma - 4V + 4 - P_f - 2P_\gamma$$

$$= 4 + 3P_f + 2P_\gamma - 3V - V$$

$$= 4 + 3P_f + 2P_\gamma - \frac{3}{2}(E_f + 2P_f) - E_\gamma - 2P_\gamma$$

$$= 4 - \frac{3}{2}E_f - E_\gamma$$

Problem 8.2 (One-loop structure of QED)

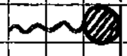
a) We are to give D for the following seven amplitudes

i) vacuum energy  $E_f = E_\gamma = 0$, $D = 4$

ii) photon propagator  $E_f = 0$, $E_\gamma = 2$, $D = 2$

iii) electron propagator  $E_f = 2$, $E_\gamma = 0$, $D = 1$

iv) electron-electron-photon vertex  $E_f = 2$, $E_\gamma = 1$, $D = 0$

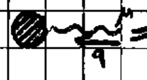
v) 1-photon amplitude  $E_f = 0$, $E_\gamma = 1$, $D = 3$

vi) 3-photon amplitude  $E_f = 0$, $E_\gamma = 3$, $D = 1$

vii) 4-photon amplitude  $E_f = 0$, $E_\gamma = 4$, $D = 0$

b) Aim: Show that one-loop photon one-point function vanishes by discrete symmetry under charge conjugation

The only way for an external photon to enter a diagram is via a QED vertex, so we may 'drag' that out of  and write

 $=$  $= -ie \int d^4x\, e^{-iqx} \langle \Omega | T j^\mu(x) | \Omega \rangle$,

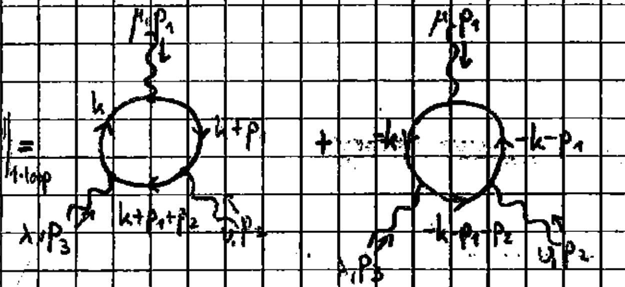
with $j^\mu(x) = \bar\Psi(x) \gamma^\mu \Psi(x)$. Since $\hat{C}$ is a symmetry of QED, i.e. $\hat{C}|\Omega\rangle = |\Omega\rangle$, but changes the sign of any current $j^\mu(x)$ the amplitude must be zero,

$$\langle \Omega | T j^\mu(x) | \Omega \rangle = \langle \Omega | \underbrace{\hat{C}^2}_{1} T j^\mu(x) \underbrace{\hat{C}^2}_{1} | \Omega \rangle = \langle \Omega | \hat{C}\, T \underbrace{\hat{C}\, j^\mu(x)\, \hat{C}}_{-j^\mu(x)}\, \hat{C} | \Omega \rangle$$

$$= -\langle \Omega | T j^\mu(x) | \Omega \rangle = 0.$$

Aim: Show that the two diagrams contributing to the photon three-point function at one-loop order cancel

The contributions to $\Gamma^{(3)}$ at one-loop order are given by fermionic loops that differ only in their direction of momentum flow. Using the fermion propagator $S_F(k) = \frac{i}{\not{k} - m}$, we write

$i\Gamma^{(3)}\big|_{1\text{-loop}} =$ 

$$= (-ie)^3 \int \frac{d^4k}{(2\pi)^4} \Big[-\mathrm{Tr}\big(S_F(k)\gamma^\mu S_F(k+p_1)\gamma^\nu S_F(k+p_1+p_2)\gamma^\lambda \big)$$
$$- \mathrm{Tr}\big(S_F(-k)\gamma^\mu S_F(-k-p_1)\gamma^\nu S_F(-k-p_1-p_2)\gamma^\lambda \big)\Big]$$

The only thing in a fermionic propagator $S_F(k)$ affected by charge conjugation is the γ-matrix. By requiring that charge conjugation of the Dirac equation for a field Ψ with charge e gives the Dirac eq. for a field Ψ^c with charge $-e$, we obtain the condition $\hat{C}\gamma^\mu\hat{C} = -(\gamma^\mu)^T$. Thus

$$\hat{C} S_F(-k)\hat{C} = \hat{C}\frac{i}{-\not{k}-m}\hat{C} = \hat{C}\frac{i(-\not{k}+m)}{k^2-m^2}\hat{C} = \frac{i((\gamma^\mu)^T k_\mu + m)}{k^2-m^2} = \frac{i(\not{k}^T+m)}{k^2-m^2} = S_F^T(k).$$

Applying this to $i\Gamma^{(3)}|_{1\text{-loop}}$, we find

$$i\Gamma^{(3)}\big|_{1\text{-loop}} = (-ie)^3 \int \frac{d^4k}{(2\pi)^4} \Big[-\mathrm{Tr}\big(S_F(k)\gamma^\mu S_F(k+p_1)\gamma^\nu S_F(k+p_1+p_2)\gamma^\lambda \big)$$
$$- \mathrm{Tr}\big(\hat{C}^2 S_F(-k)\hat{C}^2\gamma^\mu\hat{C}^2 S_F(-k-p_1)\hat{C}^2\gamma^\nu\hat{C}^2 S_F(-k-p_1-p_2)\hat{C}^2\gamma^\lambda \big)\Big]$$
$$= (-ie)^3 \int \frac{d^4k}{(2\pi)^4} \Big[-\mathrm{Tr}\big(S_F(k)\gamma^\mu S_F(k+p_1)\gamma^\nu S_F(k+p_1+p_2)\gamma^\lambda \big)$$
$$+ \mathrm{Tr}\big(S_F^T(k)(\gamma^\mu)^T S_F^T(k+p_1)(\gamma^\nu)^T S_F^T(k+p_1+p_2)(\gamma^\lambda)^T \big)\Big] = 0$$

Aim: Show that the diagrams contributing to any n-point photon amplitude, for n odd, cancel in pairs

Obviously we may employ the above scheme for any fermionic loop with an odd number of vertices.

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Problem 9.1 (Renormalization of Yukawa theory)

$$\mathcal{L}_Y = \frac{1}{2}(\partial\phi_0)^2 - \frac{1}{2} m_0^2 \phi_0^2 + \bar{\Psi}_0 (i \partial\!\!\!/ - M_0) \Psi_0 - i g_0 \phi_0 \bar{\Psi}_0 \gamma^5 \Psi_0$$

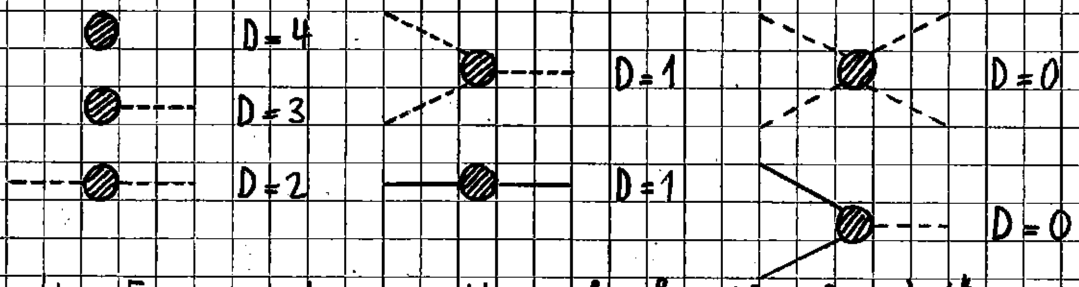
invariant under parity $\Psi(t,\vec{x}) \longrightarrow \gamma^0 \Psi(t,-\vec{x})$ and $\phi(t,\vec{x}) \longrightarrow -\phi(t,-\vec{x})$.

a) determine superficially divergent amplitudes and work out Feynman rules for renormalized perturbation

Superficial degree of divergence is the same as that of QED since Lagrangian has the same structure, only need to replace gauge field A_μ with scalar field ϕ

$$\left.\begin{array}{l} D = 4L - P_f - 2P_\phi \\ L = P_f + P_\phi - V + 1 \\ V = \frac{1}{2}(E_f + 2P_f) = E_\phi + 2P_\phi \end{array}\right\} D = 4 - \frac{3}{2} E_f - E_\phi$$

Superficially divergent amplitudes are those with $D \geq 0$:



To obtain Feynman rules, we split up $\mathcal{L} = \mathcal{L}_Y + \delta\mathcal{L}$, $\delta\mathcal{L} = -\frac{\lambda_0}{4!}\phi_0^4$

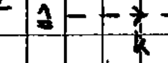
$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2} m^2 \phi^2 + \bar{\Psi}(i\partial\!\!\!/ - M)\Psi - i g \phi \bar{\Psi}\gamma^5\Psi - \frac{\lambda}{4!}\phi^4$$

$$+ \frac{1}{2}\delta_\phi (\partial\phi)^2 - \frac{1}{2}\delta_m \phi^2 + \bar{\Psi}(i\delta_\psi \partial\!\!\!/ - \delta_M)\Psi - i\delta_g \phi\bar{\Psi}\gamma^5\Psi - \frac{\delta_\lambda}{4!}\phi^4$$

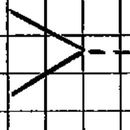
into renormalized fields, $\phi = Z_\phi^{-\frac{1}{2}}\phi_0$, and parameters m, M, g, and the counterterms,

$\delta_\phi = Z_\phi - 1$, $\delta_m = m_0 Z - m$, $\delta_\psi = Z_\psi - 1$, $\delta_M = M_0 Z - M$,

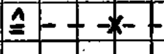
$\delta_g = g Z_\phi^{\frac{1}{2}} Z_\psi$, $\delta_\lambda = \lambda_0 Z^2 - \lambda$

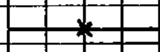
$\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 \hat{=}$  $= \frac{i}{k^2-m^2}$

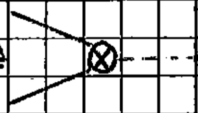
$\bar{\Psi}(i\not{\partial} - M)\Psi \hat{=}$  $= \frac{i}{\not{p}-M} = \frac{i(\not{p}+M)}{p^2-M^2}$

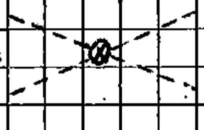
$-ig\phi\bar{\Psi}\gamma^5\Psi \hat{=}$  $= g\gamma^5$,

$-\frac{\lambda}{4!}\phi^4 \hat{=}$  $= -i\lambda$

$\frac{1}{2}\delta_\phi(\partial\phi)^2 - \frac{1}{2}\delta_m\phi^2 \hat{=}$  $= i(p^2\delta_\phi - \delta_m)$

$\bar{\Psi}(i\not{\partial} - M)\Psi \hat{=}$  $= i(\not{p}\delta_\psi - \delta_M)$

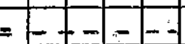
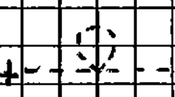
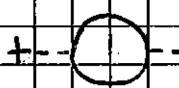
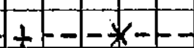
$-i\delta_g\phi\bar{\Psi}\gamma^5\Psi \hat{=}$  $= \delta_g\gamma^5$

$-\frac{\delta_\lambda}{4!}\phi^4 \hat{=}$  $= -i\delta_\lambda$

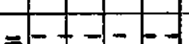
b) Aim: compute divergent part of each counterterm to the one-loop order

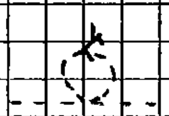
 no computation necessary, can be trivially absorbed into vacuum energy constant V_0 in the Lagrangian

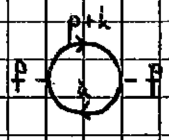
 $= 0$ by parity invariance of Lagrangian

 $\Big|_{1\text{-loop}} =$  $\underbrace{+}_{}$  $+$  $+$ 

$= 0$ at $p^2 = m^2$

$=$  $+ \, i M^2(p^2)\Big|_{1\text{-loop}}$

 $= -\frac{i}{2}\lambda \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2} = -\frac{im^2\lambda}{32\pi^2}\left(\frac{2}{\epsilon} - \gamma + 1 + \ln(4\pi)\right)$

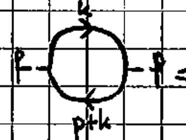
 $= -g^2 \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{Tr}(\gamma^5(\not{k}+M)\gamma^5(\not{k}+\not{p}+M))}{(k^2-M^2)((k+p)^2-M^2)}$

$\mathrm{Tr}(\gamma^5(\not{k}+M)\gamma^5(\not{k}+\not{p}+M)) = \mathrm{Tr}(-\not{k}^2 - \not{k}\not{p} + \not{p}M + M^2) = 4(M^2 - k^2 - k\cdot p)$

For the denominator, we use a Feynman parametrization

$$(k^2-M^2)^{-1}((k+p)^2-M^2)^{-1} = \int_0^1 dx \frac{1}{[x((k+p)^2-M^2)+(1-x)(k^2-M^2)]^2}$$

$$x((k+p)^2-M^2)+(1-x)(k^2-M^2) = xp^2+2xkp+k^2-M^2 = (\underbrace{k+xp}_{L})^2 \underbrace{-x(1-x)p^2-M^2}_{-\Delta}$$



$$= 4g^2\int\frac{d^dk}{(2\pi)^d}\int_0^1 dx\, \frac{k^2-M^2+kp}{(L^2-\Delta)^2} = 4g^2\int\frac{d^dL}{(2\pi)^d}\int_0^1 dx\, \frac{L^2-\overbrace{2xLp}^{0}+x^2p^2+\overbrace{Lp}^{0}-xp^2}{(L^2-\Delta)^2}$$

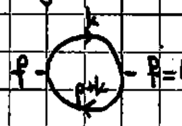
$$= 4g^2\int\frac{d^dk}{(2\pi)^d}\int_0^1 dx\, \frac{L^2-x(1-x)p^2-M^2}{(L^2-\Delta)^2}$$

Using the formulae of dimensional regularization,

$$\int\frac{d^dL}{(2\pi)^d}\frac{1}{(L^2-\Delta)^n} = \frac{i(-1)^n}{(4\pi)^{d/2}}\frac{\Gamma(n-\frac d2)}{\Gamma(n)}\Delta^{\frac d2-n}$$

$$\int\frac{d^dL}{(2\pi)^d}\frac{L^2}{(L^2-\Delta)^n} = \frac{i(-1)^{n-1}}{(4\pi)^{d/2}}\frac{\Gamma(n-\frac d2-1)}{\Gamma(n)}\Delta^{1+\frac d2-n},$$

we get



$$= 4g^2\int_0^1 dx\left(-\frac{i\Gamma(1-\frac d2)}{(4\pi)^{d/2}}\Delta^{\frac d2-1}-\frac{i\Gamma(2-\frac d2)}{(4\pi)^{d/2}}\Delta^{\frac d2-2}\left(x(1-x)p^2+M^2\right)\right)$$

$$= -\frac{4ig^2}{(4\pi)^{d/2}}\int_0^1 dx\, \Gamma(\underbrace{2-\tfrac d2}_{\epsilon/2})\Big(\underbrace{1-\frac{1}{1-\epsilon/2}}_{\frac\epsilon2+\sigma(\epsilon^2)}\Big)\Delta^{\frac d2-1}$$

$$= -\frac{4ig^2}{16\pi^2}(4\pi)^{\frac\epsilon2}\left(\frac2\epsilon-\gamma+\sigma(\epsilon)\right)\left(\frac\epsilon2+\sigma(\epsilon^2)\right)\int_0^1 dx\left(x(1-x)p^2+M^2\right)^{1-\frac\epsilon2}$$

$$= -\frac{ig^2}{4\pi^2}\left(1+\frac\epsilon2\ln(4\pi)+\sigma(\epsilon)\right)\left(\frac2\epsilon-\gamma+1+\sigma(\epsilon)\right)\int_0^1 dx\left[x(1-x)p^2+M^2\right]\left[1-\frac\epsilon2\left(x(1-x)p^2+M^2\right)+\sigma(\epsilon^2)\right]$$

$$= -\frac{ig^2}{4\pi^2}\left(\frac2\epsilon-\gamma+1+\ln(4\pi)+\sigma(\epsilon)\right)\int_0^1 dx\left[x(1-x)p^2+M^2-\frac\epsilon2\left(x(1-x)p^2+M^2\right)^2+\sigma(\epsilon^2)\right]$$

$$= -\frac{ig^2}{4\pi^2}\left(\frac2\epsilon-\gamma+1+\ln(4\pi)\right)\left[\frac{p^2}{6}+M^2-\frac\epsilon2\left(\frac{p^4}{30}+\frac{p^2}{3}M^2+M^4\right)+\sigma(\epsilon^2)\right]$$

$$= -\frac{ig^2}{4\pi^2}\left(\frac2\epsilon-\gamma+1+\ln(4\pi)\right)\left(\frac{p^2}{6}+M^2\right)-\left(\frac{p^4}{30}+\frac{p^2}{3}M^2+M^4\right)$$

Problem 10.1 (One-loop β-function in QED)

a) Aim: compute the one-loop β-function in QED

$$\beta_e(\mu) = \mu \frac{\partial}{\partial \mu}\left(-e\delta_1 + e\delta_2 + \frac{1}{2} e\,\delta_3\right)$$

$$\delta_1 = -\frac{e^2}{(4\pi)^{d/2}} \int_0^1 dz\,(1-z)\left[\frac{\Gamma(2-\frac{d}{2})}{[(1-z)^2 m^2 + z M^2]^{2-d/2}} \frac{(2-\epsilon)^2}{2}\right.$$

$$\left. + \frac{\Gamma(3-\frac{d}{2})}{[(1-z)^2 m^2 + z M^2]^{3-\frac{d}{2}}}\left(2(1-4z+z^2) - \epsilon(1-z)^2\right) m^2\right],$$

$$\delta_2 = -\frac{e^2}{(4\pi)^{d/2}} \int_0^1 dz\, \frac{\Gamma(2-\frac{d}{2})}{[(1-z)^2 m^2 + z M^2]^{2-d/2}}\left[(2-\epsilon) z\right.$$

$$\left. - \frac{\epsilon}{2} \frac{2z(1-z) m^2}{(1-z)^2 m^2 + z M^2}\left(4 - 2z - \epsilon(1-z)\right)\right],$$

$$\delta_3 = -\frac{e^2}{(4\pi)^{d/2}} \int_0^1 dz\, \frac{\Gamma(2-\frac{d}{2})}{(m^2)^{2-d/2}}\left(8z(1-z)\right),$$

where M is an IR cutoff and m is the electron mass, playing the role of the renormalization scale μ.

Since M is just an IR cutoff and QED is well-defined in the IR, we can, right from the outset, set it to zero. Thus

$$m\frac{\partial \delta_1}{\partial m} = m\frac{\partial}{\partial m}\left[-\frac{e^2}{(4\pi)^{d/2}}\int_0^1 dz\,(1-z)\left[\frac{\Gamma(\frac{\epsilon}{2})}{((1-z)m)^\epsilon}\frac{(2-\epsilon)^2}{2} + \frac{\Gamma(1+\frac{\epsilon}{2})}{(1-z)^{2+\epsilon} m^\epsilon}\left(2(1-4z+z^2) - \epsilon(1-z)^2\right)\right]\right]$$

$$= \frac{\epsilon e^2}{(4\pi)^{d/2} m^\epsilon}\int_0^1 dz\left[\frac{\frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)}{2(1-z)^{\epsilon-1}}(4 - 4\epsilon + \epsilon^2) + \frac{\frac{\epsilon}{2}\left(\frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)\right)}{(1-z)^{1+\epsilon}}\left(2(1-4z+z^2) - \epsilon(1-z)^2\right)\right]$$

$$\xrightarrow{\epsilon\to 0} \frac{e^2}{(4\pi)^2}\int_0^1 dz\; 4(1-z) = \frac{e^2}{8\pi^2}$$

$$m\frac{\partial \delta_2}{\partial m} = m\frac{\partial}{\partial m}\left(-\frac{e^2}{(4\pi)^{d/2}}\int_0^1 dz\,\frac{\Gamma(\frac{\epsilon}{2})}{((1-z)m)^\epsilon}\left[(2-\epsilon)z - \frac{\epsilon}{2}\, 2z\left(4 - 2z - \epsilon(1-z)\right)\right]\right)$$

$$= \frac{\epsilon e^2}{(4\pi)^{d/2} m^\epsilon}\int_0^1 dz\,\frac{\frac{2}{\epsilon} - \gamma + \mathcal{O}(\epsilon)}{(1-z)^\epsilon}\left[(2-\epsilon)z - \epsilon\left(4 - 2z - \epsilon(1-z)\right)\right]$$

$$\xrightarrow{\epsilon\to 0} \frac{e^2}{(4\pi)^2}\int_0^1 dz\; 2 = \frac{e^2}{8\pi^2}$$

$$m\frac{\partial \delta_3}{\partial m} = m\frac{\partial}{\partial m}\left(-\frac{e^2}{(4\pi)^{d/2}}\int_0^1 dz\,\frac{\Gamma(\frac{\epsilon}{2})}{m^{\epsilon}}\,8z(1-z)\right) = \frac{\epsilon e^2}{(4\pi)^{d/2} m^{\epsilon}}\int_0^1 dz\left(\frac{2}{\epsilon}-\gamma+\mathcal{O}(\epsilon)\right)8z(1-z)$$

$$\xrightarrow{\epsilon\to 0} \frac{e^2}{(4\pi)^2}\int_0^1 dz\;16z(1-z) = \frac{e^2}{6\pi^2}$$

Inserting $\delta_1, \delta_2, \delta_3$ into the one-loop β-function of QED, we get

$$\beta_e(m) = -e\frac{e^2}{8\pi^2} + e\frac{e^2}{8\pi^2} + \frac{1}{2}e\frac{e^2}{6\pi^2} = \frac{e^3}{12\pi^2}$$

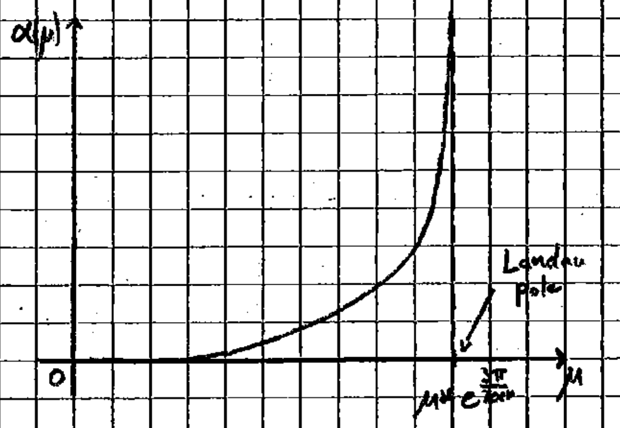
b) Aim: Integrate RG eq. to find the running of the coupling

$$\text{RG eq.:}\quad \frac{d}{d\ln\mu}e(\mu) = \beta_e(\mu) = \frac{e^3(\mu)}{12\pi^2} \implies 12\pi^2\frac{de(\mu)}{e^3(\mu)} = d\ln\mu$$

$$\int_{\mu^*}^{\mu} 12\pi^2\frac{de(\mu')}{e^3(\mu')} \underset{\substack{\uparrow\\ \alpha(\mu)=\frac{e^2(\mu)}{4\pi},\; d\alpha(\mu)=\frac{e(\mu)}{2\pi}de(\mu)}}{=} \int_{\mu^*}^{\mu} 12\pi^2\frac{2\pi}{e(\mu')}d\alpha(\mu')\frac{1}{e^3(\mu')} = \frac{3}{2}\pi\int_{\mu^*}^{\mu}\frac{d\alpha(\mu')}{\alpha^2(\mu')} = -\frac{3}{2}\pi\left(\frac{1}{\alpha(\mu)}-\frac{1}{\underbrace{\alpha(\mu^*)}_{:=\alpha^*}}\right)$$

$$= \int_{\mu^*}^{\mu} d\ln\mu' = \ln\left(\frac{\mu}{\mu^*}\right) \implies \frac{1}{\alpha(\mu)} - \frac{1}{\alpha^*} = -\frac{2}{3\pi}\ln\left(\frac{\mu}{\mu^*}\right)$$

$$\alpha(\mu) = \frac{1}{\frac{1}{\alpha^*}-\frac{2}{3\pi}\ln\left(\frac{\mu}{\mu^*}\right)} = \frac{\alpha^*}{1-\frac{2\alpha^*}{3\pi}\ln\left(\frac{\mu}{\mu^*}\right)},\quad \text{pole at } 1 = \frac{2\alpha^*}{3\pi}\ln\left(\frac{\mu}{\mu^*}\right) \Rightarrow \mu = \mu^* e^{\frac{3\pi}{2\alpha^*}}$$



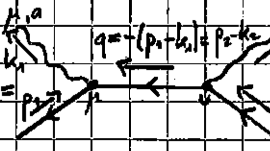
$e(\mu)$ ceases to be perturbative if it of the order $e(\mu)\approx 1$, i.e.

$$\frac{1}{4\pi} = \frac{1}{\frac{1}{\alpha^*}-\frac{2}{3\pi}\ln\left(\frac{\mu}{\mu^*}\right)} \implies \ln\left(\frac{\mu}{\mu^*}\right) = \frac{3\pi}{2}\left(\frac{1}{\alpha^*}-4\pi\right),\quad \mu = \mu^* e^{\frac{3\pi}{2}\left(\frac{1}{\alpha^*}-4\pi\right)}$$

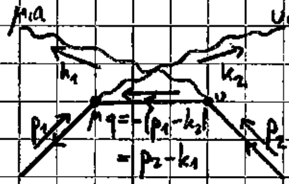
20.07.2015

Problem 11.2 (Fermion - anti-fermion annihilation in Yang-Mills theory at tree-level)

a) Aim: Use Feynman rules to compute t- and u-channel of this process

$i\mathcal{M}_1^{ab} =$ 

$= \bar v(p_1)(ig\gamma^\mu T^a)\frac{i(\slashed q + m)}{q^2-m^2}(ig\gamma^\nu T^b)u(p_2)\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)$

$i\mathcal{M}_2^{ab} =$ 

$= \bar v(p_1)(ig\gamma^\nu T^b)\frac{i(\slashed q + m)}{q^2-m}(ig\gamma^\mu T^a)u(p_2)\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)$

$$i\mathcal{M}_{1+2}^{ab} = i\mathcal{M}_1^{ab} + i\mathcal{M}_2^{ab} = -i(ig)^2\bar v(p_1)\left[\gamma^\mu T^a\frac{\slashed p_1-\slashed k_1-m}{(p_1-k_1)^2-m^2}T^b\gamma^\nu + \gamma^\nu T^b\frac{\slashed p_1-\slashed k_2-m}{(p_1-k_2)^2-m^2}\gamma^\mu T^a\right]u(p_2)\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)$$

If $\epsilon^*_\nu(k_2) = k_{2\nu}$, we may use the Dirac eq. to intelligently add zero:

$\slashed k_2 u(p_2) = -(\slashed p_2 - \slashed k_2 - m)u(p_2) = (\slashed p_1 - \slashed k_1 + m)u(p_2)$, since $-(\slashed p_2 - m)u(p_1) = 0$.

$\bar v(p_1)\slashed k_2 = -\bar v(p_1)(\slashed p_1 - \slashed k_2 + m)$, since $-\bar v(p_1)(\slashed p_1 + m) = 0$.

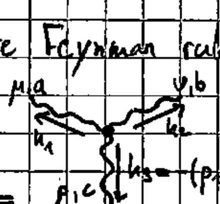
Thus, for $\epsilon^*_\nu(k_2) = k_{2\nu}$, $i\mathcal{M}_{1+2}^{ab}$ becomes

$$i\mathcal{M}_{1+2}^{ab} = -i(ig)^2\bar v(p_1)\left[\gamma^\mu T^a\frac{\slashed p_1-\slashed k_1-m}{(p_1-k_1)^2-m^2}T^b\slashed k_2 + \slashed k_2 T^b\frac{\slashed p_1-\slashed k_2-m}{(p_1-k_2)^2-m^2}T^a\gamma^\mu\right]\epsilon^*_\mu(k_1)$$

$$= -i(ig)^2\bar v(p_1)\left[\gamma^\mu T^a\frac{(p_1-k_1)^2-m^2}{(p_1-k_1)^2-m^2}T^b - T^b\frac{(p_1-k_2)^2-m^2}{(p_1-k_2)^2-m^2}T^a\gamma^\mu\right]\epsilon^*_\mu(k_1)$$

$$= (ig)^2\bar v(p_1)\left[-i\gamma^\mu[T^a, T^b]\right]u(p_2)\epsilon^*_\mu(k_1)$$

b) Aim: Use Feynman rules to compute s-channel

$i\mathcal{M}_3^{ab} =$ 

$= \bar v(p_1)(ig\gamma^\sigma T^c)u(p_2)\frac{-i\eta_{\rho\sigma}}{k_3^2}gf^{abc}\left[\eta^{\mu\nu}(k_1-k_2)^\rho + \eta^{\nu\rho}(k_2-k_3)^\mu + \eta^{\rho\mu}(k_3-k_1)^\nu\right]\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)$

$$= g^2\bar v(p_1)\gamma_\rho u(p_2)\frac{1}{k_3^2}f^{abc}T^c\left[\eta^{\mu\nu}(k_1-k_2)^\rho + \eta^{\nu\rho}(k_2-k_3)^\mu + \eta^{\rho\mu}(k_3-k_1)^\nu\right]\epsilon^*_\mu(k_1)\epsilon^*_\nu(k_2)$$

Again assuming longitudinal polarization, $\epsilon^*_\nu(k_2) = k_{2\nu}$, for one outgoing photon and transverse polarization, $\epsilon^*_\mu(k_1)\, k_1^\mu = 0$, for the other, we get

$$iM_3^{ab} = g^2 \bar v(p_1)\gamma_\rho u(p_2) \frac{1}{k_3^2} f^{abc} T^c \left[k_2^\mu (k_1-k_2)^\rho + k_2^\rho (k_2-k_3)^\mu + \eta^{\mu\rho}(k_3-k_1)\cdot k_2\right] \epsilon^*_\mu(k_1),$$

$$\left[k_2^\mu(k_1-k_2)^\rho + k_2^\rho(k_2-k_3)^\mu + \eta^{\mu\rho}(k_3-k_1)\cdot k_2\right]\epsilon^*_\mu(k_1)$$

$$= \left[-(k_1+k_3)^\mu(2k_1+k_3)^\rho + (k_1+k_3)^\rho(k_1+2k_3)^\mu - \eta^{\mu\rho}(k_3-k_1)\cdot(k_1+k_3)\right]\epsilon^*_\mu(k_1)$$

$$\overset{k_2=-k_1-k_3}{=} \Big[-2k_1^\mu k_1^\rho - \cancel{k_1^\mu k_3^\rho} - \cancel{2k_1^\rho k_3^\mu} - k_3^\mu k_3^\rho + k_1^\mu k_1^\rho + \cancel{2k_1^\rho k_3^\mu} + \cancel{k_1^\mu k_3^\rho} + 2k_3^\mu k_3^\rho$$
$$+ \eta^{\mu\rho}\big(\underbrace{k_1^2}_{0} - k_3^2\big)\Big]\epsilon^*_\mu(k_1)$$

$$= \left[-k_1^\mu k_1^\rho + k_3^\mu k_3^\rho - \eta^{\mu\rho}k_3^2\right]\epsilon^*_\mu(k_1) = \left(k_3^\mu k_3^\rho - \eta^{\mu\rho}k_3^2\right)\epsilon^*_\mu(k_1)$$

$$\Longrightarrow iM_3^{ab} = g^2 \bar v(p_1)\gamma_\rho u(p_2)\frac{1}{k_3^2} f^{abc}T^c\left(k_3^\mu k_3^\rho - \eta^{\mu\rho}k_3^2\right)\epsilon^*_\mu(k_1)$$

$$= -g^2 \bar v(p_1)\gamma^\mu u(p_2) f^{abc}T^c\epsilon^*_\mu(k_1),$$

where the first term vanishes because

$$\bar v(p_1)\gamma_\rho u(p_2)k_3^\rho = \bar v(p_1)\not k_3 u(p_2) = \bar v(p_1)\big(\underbrace{\not k_3 + \not p_1}_{-\not p_2} - \not p_1 - m + m\big)u(p_2)$$

$$= \underbrace{\bar v(p_1)(-\not p_1 - m)}_{0} u(p_2) + \bar v(p_1)\underbrace{(-\not p_2 + m)\,u(p_2)}_{0} = 0.$$

c) Aim: Compute iM_3^{ab} for $\epsilon^*_\mu(k_1)=\epsilon^+_\mu(k_1)$, $\epsilon^*_\nu(k_2)=\epsilon^-_\nu(k_2)$ with $\epsilon^\pm(k) = \frac{1}{\sqrt{2}|\vec k|}(k^0, \mp\vec k)$

In this setup, the polarization vector $\epsilon^*_\nu(k_2)$ is still longitudinal, just of different length than in part b). We may thus quote our result from b) as it was just before using that $\epsilon^*_\mu(k_1)$ is transverse,

$$iM_3^{ab} = g^2\bar v(p_1)\gamma_\rho u(p_2)\frac{1}{k_3^2}f^{abc}T^c\left[-k_1^\mu k_1^\rho + k_3^\mu k_3^\rho - \eta^{\mu\rho}k_3^2\right]\frac{1}{\sqrt{2}|\vec k_2|}\epsilon^+_\mu(k_1)$$

$$= \frac{1}{\sqrt{2}|\vec k_2|}\, iM^{ab}_{3,b)}\Big|_{\epsilon^*_\mu(k_1)=\epsilon^+_\mu(k_1)} - g^2\bar v(p_1)\gamma_\rho u(p_2)\frac{1}{k_3^2}f^{abc}T^c \underbrace{\frac{k_1^\mu k_1^\rho}{\sqrt{2}|\vec k_2|}\frac{k_{1\mu}}{\sqrt{2}|\vec k_1|}}_{k_1^\rho |\vec k_1|/|\vec k_2|,\ \text{since } k_1^2=0},$$

where $k_3^\mu k_3^\rho$-term again vanishes by the same argument and the $-\eta^{\mu\rho}k_3^2$-term gives just a scaled version of the result of part b).